# COMBING, LOCAL-TO-GLOBAL PROPERTIES FOR MORSE QUASI-GEODESICS AND DIVERGENCE

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Abstract. In this paper we show that a space with a quasi-consistent bounded quasi-geodesic bicombing satisfies the Morse local-to-global property. The techniques developed allow us to show that if such has a cut-point in an asymptotic cone, then it needs to have one in all of them. So far, the only other class known to satisfy this dichotomy and containing groups with and without cut-points in their asymptotic cones was CAT(0) spaces.

## 1. Introduction

This paper is concerned with local-to-global properties of Morse quasi-geodesics in spaces that admit (bi)combings. Roughly speaking, a quasi-geodesic is Morse if it is "the only way" to connect two points: any other quasi-geodesic connecting the same points needs to be close to the first quasi-geodesic, that is to say, they behave as geodesics in Gromov hyperbolic spaces. Morse quasi-geodesics are a central object in the program of classifying groups up to quasi-isometry: quasi-isometries take Morse quasi-geodesics to Morse quasi-geodesic, and thus they are a natural object to study to establish which properties the two quasi-isometric groups have in common. For instance, they can be organized in boundaries, an avenue of study that proved very fruitful [4–6, 24, 28].

As mentioned, there is a close relation between the Morse property and hyperbolicity, for instance a group is hyperbolic if an only if all of its quasi-geodesics are Morse [4, Section 10]. It is therefore very tempting to assume that theorems about geodesics in hyperbolic groups hold for Morse quasi-geodesics in a general group. Unfortunately, this is not the case. Ol'shanskii, Osin, and Sapir constructed a torsion free, finitely generated, non-virtually cyclic group $G$ where *all* periodic quasi-geodesics are Morse but $G$ does not have non-abelian free group, showing that even a requirement as strong as asking that all periodic quasi-geodesic are Morse is in general not sufficient to apply the ping-pong Lemma [22]. However, the group $G$ above is rather pathological, and in many examples of interest Morse geodesics do behave as expected. To make this sentiment precise, Russel, Tran and the second author introduced the *Morse local-to-global* (MLTG) in [25] in order to isolate a property that on one hand allows to get much stronger control about Morse geodesics, and on the other is permissive enough to be satisfied by a very large class of groups.

**Definition 1.1.** *A metric space $X$ satisfies the* Morse local-to-global (MLTG) *property if for every Morse gauge $M$ and quasi-geodesic constants $(\lambda, \kappa)$ there is a scale $L$ so that if a path is $L$–locally an $M$–Morse $(\lambda, \kappa)$–quasi-geodesic, then it is*

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*globally a Morse quasi geodesic, where the global constants only depend on the local ones.*

Even without knowing what a Morse quasi-geodesic is, for which we refer to Definition 2.10, it is not hard to get a feeling of the above property, as it requires that paths that are "good" at a large enough scale, are still good. Local-to-global properties have been previously studied in geometric group theory, perahps the two most known example are the Cartan-Hadamard theorem and the fact that a space where local quasi-geodesics are global quasi-geodesics is a Gromov hyperbolic space [18].

1.1. **Why consider the Morse local-to-global property.** The MLTG property has many consequences of interests regarding the behaviour of Morse geodesics. For a group $G$ with the MLTG property the following holds. If $H, K \leq G$ are *stable subgroups*, there are conditions analogous the hyperbolic case that guarantee $\langle H, K\rangle \cong H *_{H\cap K} K$, i.e. the group they generate is as large as possible. The translation lengths of conjugacy classes of Morse elements on a Cayley graph form a discrete set of rational numbers. If $G$ contains a Morse element and $N$ is infinite and normal in $G$, then $N$ contains a Morse element [25]. There is a growth gap between the growth of $G$ and any infinite index stable subgroup $H$ of $G$. Moreover, the growth series of $H$ is rational, and the language of geodesics between elements of $H$ is regular. In general, the set of Morse geodesics (for fixed parameters) form a regular language [8]. If $G$ has the property that stable subgroups are separable, then the product of finitely many stable subgroups is separable [21]. The Morse boundary of $G$ is strongly $\sigma$–compact, and if $G$ is not hyperbolic and satisfy a mild non-positive curvature condition, the image of it needs to have vanishing measure in any boundary equipped with a stationary measure [20].

1.2. **Main results.** Thus, it is natural to wonder which groups have the MLTG property. Our main result is the following.

**Theorem 1.2** (Morse local-to-global)**.** *Let $X$ be a metric space with a bounded, quasi-consistent bicombing. Then $X$ satisfies the MLTG property.*

Previously known examples of groups with the MLTG property are hyperbolic groups, Morse limited groups such as solvable groups, CAT(0) groups, fundamental groups of closed three manifolds [25] and injective groups [27].

The key ingredient in the proof of Theorem 1.2 is to use the bicombing to translate between scales. Morally, if the path fails to be globally Morse, then one can transport and shrink this fact to the scale where the path is indeed Morse to get a contradiction. This is done via using the bicombing to construct paths that avoid a ball but have length comparable to the radius of the ball. In particular, this allows us to get a much stricter control on divergence.

**Theorem 1.3.** *Let $X$ be a geodesic metric space equipped with a bounded quasi-consistent $(\lambda_0, \kappa_0)$–bicombing. Assume that for every point $x \in X$ the ball $B(x, \kappa_0)$ intersects a bi-infinite $(\lambda_0, \kappa_0)$-quasi-geodesic from the bicombing (i.e. with $k_0$-tubular neighbourhood containing longer and longer bicombing lines). If there exists a sequence $n_k$ diverging to infinity such that on that sequence the divergence is bounded by a linear function then the divergence function is bounded by a linear function for every value.*

One of the most important things in the study of the divergence is to compare it with linear functions. In [11] it is proved that a groups has superlinear divergence if and only if all its asymptotic cones have global cut-points. The importance of having cut-points in asymptotic cones has been emphasized in [13] and [14]. In particular it is proved in [13] that if a non-virtually cyclic finitely generated group $G$ has cut-points in one of its asymptotic cones, then a direct power of $G$ contains a free non-Abelian subgroup. Note that there are finitely generated groups whose divergence is not linear but is arbitrarily close to being linear (and in fact is bounded by a linear function on arbitrary long intervals) [22].

This type of result was previously known, under appropriate assumptions, for very few quasi-isometry invariants (the Dehn function and the growth function).

1.3. **Weak Morse local-to-global property.** Being Morse is a condition that govern how a certain fixed quasi-geodesic behaves with respect to all other quasi-geodesics with endpoints on it. However, one might relax the condition, and consider quasi-geodesics that are Morse only with respect to quasi-geodesics *with fixed quasi-geodesic constants*. This lead to the definition of *weak Morse property*, where one needs to verify Morseness only with respect to a single constant.

Then, one can recast Theorem 1.2 in a different light by relaxing the hypotheses on the bicombing. For this, we define a *weak Morse local-to-global* (WMLTG) space to be a space where paths that are locally Morse quasi-geodesics are globally weakly Morse quasi-geodesics.

**Theorem 1.4** (Weak Morse local-to-global)**.** *Let $X$ be a metric space with a bounded, quasi-geodesic combing. Then $X$ satisfies the weak MLTG property.*

An immediate consequence, Theorem 1.4 implies that in a space with a bounded combing locally Morse quasi-geodesic are global quasi-geodesics, which is a result of independent interest. This result cannot be far from optimal as there are spaces with bounded quasi-geodesic combings where local quasi-geodesics are not global quasi-geodesics, for instance non-hyperbolic CAT(0) spaces.

The main reason to introduce the WMLTG property is motivated by the following result, that states that undes topological conditions on the Morse boundary, the weak MLTG property implies the strong one.

**Theorem 1.5.** *Let $X$ be a geodesic metric space satisfying the weak MLTG property and whose isometry group acts coboundedly on $X$. If the Morse boundary of $X$ is $\sigma$–compact then $X$ satisfies the MLTG property.*

Since He, the second author and the third author [20] showed that a space with the MLTG property needs to have a $\sigma$–compact Morse boundary, we obtain the following dichotomy.

**Corollary 1.6.** *Let $X$ be a space satisfying the weak MLTG property. Then $X$ satisfies the MLTG property if and only if the Morse boundary of $X$ is $\sigma$–compact.*

In general, understanding the topolgy of boundaries is a notoriously hard question. For instance, only recently the third author constructed the first example of a group with non-$\sigma$–compact Morse boundary using small cancellation theory [29].

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## 2. Preliminaries

Throughout the paper, unless otherwise stated, all metric spaces are geodesic. Let $X$ be a metric space.

**Definition 2.1.** *A* $(\lambda, \kappa)$–quasi-geodesic in $X$*, for* $\lambda \geq 1$ *and* $\kappa \geq 0$*, is a map* $\gamma\colon I \to X$*, where* $I \subseteq \mathbb{R}$ *is an interval, such that*

$$\frac{1}{\lambda}|t_1 - t_2| - \kappa \leq \mathrm{d}(\gamma(t_1), \gamma(t_2)) \leq \lambda|t_1 - t_2| + \kappa.$$

*We call a pair of constants* $(\lambda, \kappa)$*, with* $\lambda \geq 1$ *and* $\kappa \geq 0$*, a* quasi-geodesic pair of constants*, or simply a* quasi-geodesic pair*.*

Given $r > 0$, the *$r$–neighbourhood* of a subset $A$, i.e. $\{x \in X : \mathrm{d}(x, A) \leq r\}$, is denoted by $\mathcal{N}_r(A)$. In particular, if $A = \{a\}$ then $\mathcal{N}_r(A) = B(a, r)$ is the *closed $r$–ball centered at $a$.*

Consider a path $\mathfrak{p}\colon [a, b] \to X$.

**Notation 2.2.** *Given a point* $x \in X$ *and a subset* $A \subseteq X$*, we write, by abuse of notation,* $x \in \mathfrak{p}$ *to mean that* $x$ *belongs to the image of* $\mathfrak{p}$*, and* $\mathfrak{p} \subseteq A$ *to mean that the image of* $\mathfrak{p}$ *is contained in* $A$*.*

We denote by $\mathfrak{p}^{-1}$ the path $\mathfrak{p}^{-1} : [a, b] \to X$ where $\mathfrak{p}^{-1}(t) = \mathfrak{p}(b + a - t)$. We denote the length of a rectifiable path $\mathfrak{p}$ by $\mathrm{length}(\mathfrak{p})$ and the length of its domain, $|b - a|$, by $\mathrm{Dlength}(\mathfrak{p})$. Let $u, v \in \mathfrak{p}$. We say that $x \in \mathfrak{p}$ *lies between* $u$ and $v$ if there are $s_1 \leq s_2 \leq s_3$ in the domain of $\mathfrak{p}$ such that $\mathfrak{p}(s_1) = u$, $\mathfrak{p}(s_2) = x$, $\mathfrak{p}(s_3) = v$. If $s_1$ is the smallest parameter so that $\mathfrak{p}(s_1) = u$ and $s_3$ is the largest parameter so that $\mathfrak{p}(s_3) = v$ we denote by $\mathfrak{p}|_{uv}$ the restriction of $\mathfrak{p}$ to $[s_1, s_3]$ (i.e. the maximal sub-path of $\mathfrak{p}$ with image composed of points that lie between $u$ and $v$) and we call it *the subarc determined by $u$ and $v$.*

**Lemma 2.3** (Improved quasi-geodesics [2], Lemma 1.11, [3], Proposition 8.3.4)**.** *Let* $X$ *be a geodesic metric space. For every* $(\lambda, \kappa)$*-quasi-geodesic* $\gamma : [a, b] \to X$ *there exists a continuous* $(\lambda, \kappa')$*-quasi-geodesic* $\bar{\gamma} : [a, b] \to X$ *such that*

1. $\gamma(a) = \bar{\gamma}(a)$ *and* $\gamma(b) = \bar{\gamma}(b)$*;*
2. $\kappa' = 2(\lambda + \kappa)$*;*
3. $\mathrm{length}(\bar{\gamma}|_{[t,t']}) \leq k_1 \mathrm{d}(\bar{\gamma}(t), \bar{\gamma}(t')) + k_2$ *for all* $t, t' \in [a, b]$*, where* $k_1 = \lambda(\lambda + \kappa)$ *and* $k_2 = (\lambda\kappa' + 3)(\lambda + \kappa)$*;*
4. *the Hausdorff distance between the images of* $\gamma$ *and* $\bar{\gamma}$ *is less than* $\lambda + \kappa$*.*

We call such a quasi-geodesic $\bar{\gamma}$ an *improvement* of $\gamma$. A $(\lambda, \kappa)$–quasi-geodesic is *improved* if it is continuous and condition (3) of Lemma 2.3 holds.

**Convention 2.4.** *From now on, all quasi-geodesics are assumed to be improved.*

### 2.1. Quasi-geodesic combings.

Consider a metric space $X$ and two constants $\lambda_0 \geq 1$ and $\kappa_0 \geq 0$. A $(\lambda_0, \kappa_0)$*–quasi-geodesic (bi)-combing* is a way of assigning to every ordered pair of points $(x, y) \in X \times X$, a $(\lambda_0, \kappa_0)$–quasi-geodesic $\mathfrak{q}_{xy}$ connecting them. The quasi-geodesics $\mathfrak{q}_{xy}$ are called *combing lines.*

When needed, we assume the quasi-geodesics to be extended to $\mathbb{R}$ by constant maps.

We say that a $(\lambda_0, \kappa_0)$–quasi-geodesic combing is *bounded* if

$$\mathrm{d}(\mathfrak{q}_{xy_1}(t), \mathfrak{q}_{xy_2}(t)) \leq \kappa_0 \mathrm{d}(y_1, y_2) + \kappa_0, \tag{2.5}$$

for all $t \in \mathbb{R}$ and $x, y_1, y_2$ in $X$.

Similarly, a $(\lambda_0, \kappa_0)$–quasi-geodesic bicombing is *bounded* if

$$\mathrm{d}(\mathfrak{q}_{x_1y_1}(t), \mathfrak{q}_{x_2y_2}(t)) \leq \kappa_0 \left[\mathrm{d}(x_1, x_2) + \mathrm{d}(y_1, y_2)\right] + \kappa_0, \tag{2.6}$$

for all $t \in \mathbb{R}$ and $(x_1, y_1), (x_2, y_2)$ in $X \times X$.

A $(\lambda_0, \kappa_0)$–quasi-geodesic bicombing is *reversible* if

$$\mathfrak{q}_{xy} = \mathfrak{q}_{yx}^{-1} \tag{2.7}$$

for all $x, y$ in $X$.

Given a combing line $\mathfrak{q}_{xy}$, we denote its constant speed reparameterization by $\hat{\mathfrak{q}}_{xy} \colon [0,1] \to X$. A $(\lambda_0, \kappa_0)$–quasi-geodesic bicombing is called *quasi-conical* if for all $(x_1, y_1), (x_2, y_2)$ in $X \times X$ and $t \in [0,1]$ we have:

$$\mathrm{d}(\hat{\mathfrak{q}}_{x_1y_1}(t), \hat{\mathfrak{q}}_{x_2y_2}(t)) \leq \lambda_0 \left[(1-t)\mathrm{d}(x_1, y_1) + t\mathrm{d}(x_2, y_2)\right] + \kappa_0.$$

A $(\lambda_0, \kappa_0)$–quasi-geodesic bicombing is called *quasi-consistent* if for any combing line $\mathfrak{q}_{xy}$ and points $z_1, z_2$ in the image of it, we have

$$\mathrm{d}_{\mathrm{Haus}}(\mathfrak{q}_{xy}|_{z_1z_2}, \mathfrak{q}_{z_1z_2}) \leq \kappa_0.$$

2.2. **Morse properties.** We start with the definition of Morse (quasi-)geodesics, the central object of the paper. Intuitively, a Morse quasi-geodesic is a geodesic that is the best way to travel in a certain direction, evidenced by the fact that any other quasi-geodesic in the same direction cannot deviate too much from it. More precisely, we have the following definitions.

**Definition 2.8.** *Let $(Q, q)$ be a quasi-geodesic pair and let $\mu \geq 0$. A quasi-geodesic $\gamma$ is $(Q, q, \mu)$–weakly Morse if any $(Q, q)$–quasi-geodesic $\eta$ with endpoints $\gamma(s)$ and $\gamma(t)$ satisfies*

$$\eta \subseteq N_\mu(\gamma|_{[s,t]}).$$

A stronger version of the Morse property is formulated below. In order to define it, we need the notion of Morse gauge as introduced in [9].

**Definition 2.9** (Morse gauge)**.** *A* Morse gauge *is a function $M \colon \mathbb{R}_{\geq 1} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ that is non-decreasing in each of the two variables, and is continuous in the second variable.*

As shown in [9], the continuity in the second variable ensures that the $M$–Morse stratum $\partial_*^M X$ is compact in the Morse boundary for all Morse gauges $M$.

**Definition 2.10** (Morse quasi-geodesic)**.** *A quasi-geodesic $\gamma$ is $M$–*Morse*, for a Morse gauge $M$, if any $(Q, q)$–quasi-geodesic $\eta$ with endpoints $\gamma(s)$ and $\gamma(t)$ satisfies*

$$\eta \subseteq N_{M(Q,q)}(\gamma|_{[s,t]}).$$

The goal of this paper is to determine when and to what extent local-to-global properties hold for Morse quasi-geodesics, in the sense of the following definition, first formulated in [25].

**Definition 2.11.** *We say that a path* $\mathfrak{p}\colon [a,b]\to X$ satisfies a property $(P)$ at scale $L$ *(or that* $\mathfrak{p}$ $L$–locally satisfies $(P)$*, or* $\mathfrak{p}$ is $L$–locally $(P)$*) if for every* $t_1,t_2\in[a,b]$ *with* $|t_1-t_2|\le L$ *the restriction* $\mathfrak{p}|_{[t_1,t_2]}$ *has the property* $(P)$*. The quantity* $L$ *is called the* scale*. We say that a path* is locally $(P)$ *when it is* $L$*–locally* $(P)$ *for some scale* $L$*.*

A "local-to-global property" is a condition requiring that a path that is $L$–locally $(P)$, for $L$ large enough, is globally $(P')$.

The two properties that we consider here are the Morse property (in both standard and weak form) and the property of being a quasi-geodesic.

**Example 2.12.** CAT(0) spaces and more generally Busemann spaces (i.e. geodesic metric spaces with convex distance function) satisfy the local-to-global property for geodesics: any path that is $\epsilon$–locally a geodesic, for some $\epsilon>0$, is a geodesic. This is not the case for CAT(k) spaces, with k> 0, as for instance in the sphere $S^2$ a maximal circle is locally a geodesic at any scale smaller than the diameter.

When we consider quasi-geodesics, the situation is even more restrictive. Indeed, Gromov proved that a geodesic metric space has the local-to-global property for all its quasi-geodesics if and only if it is hyperbolic [17].

**Definition 2.13.** *A metric space* $X$ *satisfies the* Morse local-to-global *(*MLTG *for short)* property *if the following holds. For any quasi-geodesic pair* $(\lambda,\kappa)$ *and Morse gauge* $M$ *there exists a scale* $L$*, a quasi-geodesic pair* $(\lambda',\kappa')$ *and a Morse gauge* $M'$ *such that every path that is* $L$*–locally an* $M$*–Morse* $(\lambda,\kappa)$*–quasi geodesic is globally an* $M'$*–Morse,* $(\lambda',\kappa')$*–quasi-geodesic.*

The following weakening of the MLTG property also turns out to be relevant.

**Definition 2.14** (Weak MLTG)**.** *A metric space* $X$ *satisfies the* weak Morse local-to-global *(*WMLTG *for short)* property *if the following holds. For every quasi-geodesic pairs* $(\lambda,\kappa)$ *and* $(Q,q)$*, and every Morse gauge* $M$ *there exists a scale* $L$*, a quasi-geodesic pair* $(\lambda',\kappa')$ *and a constant* $\mu\ge 0$ *such that every path that is* $L$*–locally an* $M$*–Morse* $(\lambda,\kappa)$*–quasi geodesic is globally a* $(\lambda',\kappa')$*–quasi-geodesic that is* $(Q,q,\mu)$*–weakly Morse.*

The difference between the MLTG property and the WMLTG property is that the latter allows to say things globally only about being Morse *with respect to a fixed quasi-geodesic pair* $(Q,q)$.

## 2.3. Divergence.

The topics of Morse geodesics and acylindrically hyperbolic groups are closely related to those of divergence functions and to the existence of cut-points in asymptotic cones. We recall the definitions of the different types of divergence functions, as introduced in [11, §3]. They all measure the length of minimal paths joining two points while staying away from a ball around a third point.

We consider the usual relation on the set of functions $\mathbb{R}_+\to\mathbb{R}_+$, $f\preceq_C g$ if and only if

$$f(n)\le Cg(Cn)+Cn+C$$

for some $C>1$ and all $x$. This defines an equivalence relation on the set of functions $\mathbb{R}_+\to\mathbb{R}_+$, $f\equiv_C g$ if $f\preceq_C g$ and $g\preceq_C f$ that is very relevant when studying invariance up to quasi-isometry.

When there is no risk of confusion, we no longer mention the constant $C$ in the index and instead simply write $f\preceq g$ and $f\equiv g$.

Let $(X, \mathrm{d})$ be a geodesic metric space, and let $0 < \delta < 1$. Let $a, b, c \in X$ with $\mathrm{d}(c, \{a, b\}) = r > 0$, where $\mathrm{d}(c, \{a, b\})$ is the minimum of $\mathrm{d}(c, a)$ and $\mathrm{d}(c, b)$.

We define $\mathrm{div}(a, b, c; \delta)$ as the infimum of the lengths of paths connecting $a, b$ and avoiding the ball $\mathrm{B}(c, \delta r)$ (note that by definition a ball of non-positive radius is empty). If no such path exists, take $\mathrm{div}(a, b, c; \delta) = \infty$.

This function is invariant by quasi-isometries in the following sense: given an $(L, C)$-quasi-isometry between two geodesic metric spaces, $q : X \to Y$, for any three points $a, b, c$ in $X$,

$$2L\mathrm{div}(a, b, c; \delta) + C \geq \mathrm{div}(q(a), q(b), q(c); 2L\delta) \geq \frac{1}{2L}\mathrm{div}(a, b, c; \delta) - C \,. \tag{2.15}$$

The *divergence function* $\mathrm{Div}(n, \delta)$ of the space $X$ is defined as the supremum of all numbers $\mathrm{div}(a, b, c; \delta)$ with $\mathrm{d}(a, b) \leq n$.

Let $\lambda \geq 2$ and $n_0 \geq 0$. The *small divergence function* $\mathrm{div}(n; \lambda, \delta)$ is defined as the supremum of all numbers $\mathrm{div}(a, b, c; \delta)$ with $n_0 \leq \mathrm{d}(a, b) \leq n$ and

$$\lambda \mathrm{d}(c, \{a, b\}) \geq \mathrm{d}(a, b). \tag{2.16}$$

Clearly $\mathrm{div}(n; \lambda, \delta) \leq \mathrm{Div}(n; \delta)$, for every $n \geq n_0, \lambda, \delta$.

Two more divergence functions, restricting the choice of $c$, can be defined. For every pair of points $a, b \in X$, we choose and fix a geodesic $[a, b]$ joining them. The definitions of the restricted divergence functions do not depend in a significant way on the choice of the geodesic $[a, b]$. We say that a point $c$ is *between* $a$ and $b$ if $c$ is on the fixed geodesic segment $[a, b]$.

We define $\mathrm{Div}'(n; \delta)$ and $\mathrm{div}'(n; \lambda, \delta)$ in the same way as Div and div, but restricting $c$ to the set of points between $a$ and $b$. Clearly $\mathrm{Div}'(n; \delta) \leq \mathrm{Div}(n; \delta)$ and $\mathrm{div}'(n; \lambda, \delta) \leq \mathrm{div}(n; \lambda, \delta)$ for every $\lambda, \delta$.

The divergence functions and the restricted ones are $\equiv$-equivalent. This also implies that, the functions $\mathrm{Div}'$ and $\mathrm{div}'$ do not depend on the choice of geodesics, up to the equivalent relation $\equiv$.

**Lemma 2.17** ([11], Lemmata 3.10 and 3.11)**.**

*(1) For every $a, b \in X$, every $\delta \in (0, 1)$ and every $\lambda \geq 2$, we have*

$$\begin{aligned} \sup_{\substack{c \in [a,b] \\ \lambda \mathrm{d}(c,\{a,b\}) \geq \mathrm{d}(a,b)}} \mathrm{div}(a, b, c; \delta/3) &\leq \sup_{\lambda \mathrm{d}(c,\{a,b\}) \geq \mathrm{d}(a,b)} \mathrm{div}(a, b, c; \delta/3) \\ &\leq \sup_{\substack{c \in [a,b] \\ 2\lambda \mathrm{d}(c,\{a,b\}) \geq \mathrm{d}(a,b)}} \mathrm{div}(a, b, c; \delta) + \mathrm{d}(a, b). \end{aligned} \tag{2.18}$$

*(2) The same three inequalities are satisfied if we remove the condition $\lambda \mathrm{d}(c, \{a, b\}) \geq \mathrm{d}(a, b)$.*

The following sums up the other main properties of divergence functions proven in [11, §3].

**Proposition 2.19.** *Assume $X$ is one-ended, admits a co-compact action by isometries of a group and every point is at distance less than $\kappa$ from a bi-infinite quasi-geodesic. Then the following properties hold.*

(1) *The function* $\mathrm{Div}(n,\delta)$ *takes only finite values for* $\delta$ *sufficiently small. In particular, if* $X$ *is a Cayley graph of a finitely generated one-ended group, this holds for* $\delta = \frac{1}{3}$.
(2) $\mathrm{Div}(n;\delta) \leq \mathrm{div}(n;2,\delta) + 2n + O(1)$.
(3) *The function* $\mathrm{Div}(n;\delta)$ *is* $\equiv$*-equivalent to the function* $\mathrm{div}'(n;2,\delta)$.

In [11, §3] it is also proven that the divergence functions as defined above are $\equiv$-equivalent in an appropriate sense to the version of divergence function defined by S. Gersten ([16], [15]), and used to study Haken manifolds.

2.4. **Divergence and asymptotic cones.** For the notion of asymptotic cone, we refer to M. Gromov [19], further properties and open questions can be found in [10] and in [13].

Given a metric space $(X,\mathrm{d})$, a sequence $(o_n)$ of basepoints in it, and $(d_n)$ a sequence of positive numbers diverging to $\infty$, when building the corresponding asymptotic cone, the goal is to build a complete metric space that appears as limit of the sequence of rescaled metric spaces $(X, \frac{1}{d_n}\mathrm{d})$ with basepoints $o_n$. This is done *via* the choice of a non-principal ultrafilter which, in some sense, selects a converging subsequence from the given sequence. The limit thus obtained is denoted by $\mathrm{Con}^{\omega}(X,(o_n),(d_n))$.

In what follows, by cut-point we always mean global cut-point.

Recall that, with the terminology of [11], a metric space $X$ is called *wide* if none of its asymptotic cones has a cut-point; it is called *unconstricted* if one of its asymptotic cones does not have cut-points.

**Proposition 2.20** ([11], Lemma 3.17)**.** *Let* $X$ *be as in Proposition 2.19. The following areb equivalent:*

(1) $X$ *is wide;*
(2) $\mathrm{Div}(n;1/4)$ *is bounded by a linear function;*
(3) $\mathrm{Div}(n;\delta)$ *is bounded by a linear function for every* $\delta \leq 1/4$*;*
(4) $\mathrm{div}(n;2,1/4)$ *is bounded by a linear function;*
(5) $\mathrm{div}(n;\lambda,\delta)$ *is bounded by a linear function for some* $\lambda$ *and some* $\delta \leq 1/4$.

For a more detailed discussion relating existence of cut-points, divergence and existence of Morse geodesics we refer to [11].

2.5. **Middle recurrence.** Often, it will be convenient to work with a characterization of the Morse property, introduced in [11, Proposition 3.24] and [12, Proposition 1] and further developed in [1], that we explain below.

**Definition 2.21** ($\mathbf{t}$–middle)**.** *Let* $\gamma$ *be a quasi-geodesic and* $a,b \in \gamma$*. For* $\mathbf{t} \in \left(0,\frac{1}{2}\right)$*, the* $\mathbf{t}$–middle *of* $\gamma|_{ab}$ *is the set of* $x \in \gamma$ *lying between* $a,b$ *such that* $\min\{\mathrm{d}(x,a),\mathrm{d}(x,b)\} \geq \mathbf{t}\cdot\mathrm{d}(a,b)$*. We denote the* $\mathbf{t}$*–middle of* $\gamma|_{ab}$ *as* $\gamma|_{\mathbf{t}\cdot ab}$*. When* $a,b$ *are the endpoints of* $\gamma$*, we denote the* $\mathbf{t}$*–middle simply by* $\gamma|_{\mathbf{t}}$*.*

**Definition 2.22.** *Let* $\gamma$ *be a quasi-geodesic and* $\mathbf{t} \in \left(0,\frac{1}{2}\right)$*. We say that the quasi-geodesic* $\gamma$ *is* $\mathbf{t}$–middle recurrent *if there is a function* $\mathfrak{m}_{\mathbf{t}}\colon \mathbb{R}_+ \to \mathbb{R}_+$ *so that any path* $\mathfrak{p}$ *with endpoints* $a,b \in \gamma$ *and* $\mathrm{length}(\mathfrak{p}) \leq c\cdot\mathrm{d}(a,b)$ *satisfies*

$$\mathfrak{p} \cap \mathcal{N}_{\mathfrak{m}_{\mathbf{t}}(c)}(\gamma|_{\mathbf{t}\cdot ab}) \neq \emptyset.$$

*The function* $\mathfrak{m}_{\mathbf{t}}$ *is called the* $\mathbf{t}$–recurrence function *of the path* $\gamma$.

*We say that a quasi-geodesic* $\gamma$ *is* middle recurrent *if it is* $\mathbf{t}$–middle recurrent *for some fixed* $\mathbf{t} \in \left(0,\frac{1}{2}\right)$.

Note that subpaths of a $\mathbf{t}$–middle recurrent path are $\mathbf{t}$–middle recurrent with respect to the same recurrence function.

**Theorem 2.23** ([1, 11, 12])**.** *Let $\gamma$ be a quasi-geodesic in a geodesic metric space $X$. Then $\gamma$ is Morse if and only if $\gamma$ is middle recurrent. Moreover, its recurrence function can be bounded from above only in terms of its Morse gauge, and vice versa.*

The following lemma is a variation of the theorem above which we will use in the proof of 6.6.

**Lemma 2.24.** *Let $M$ be a Morse gauge and let $(\lambda, \kappa)$ be a quasi-geodesic pair. Let $\chi, \sigma, \delta$ be linear functions. There exist two constants $D \leq \ell$ such that the following holds. If $\gamma$ is a $(\lambda, \kappa)$–quasi-geodesic which is $M$–Morse, then there does not exist a path $\mathfrak{p}$ with endpoints $\gamma(t_1)$ and $\gamma(t_2)$ satisfying the following:*

*(1)* $\ell \leq t_2 - t_1 \leq \sigma(\ell)$,
*(2)* $\text{length}(\mathfrak{p}) \leq \chi(\ell)$,
*(3)* $\text{d}(\mathfrak{p}, \gamma|_{[t_1+\delta(D), t_1+\ell-\delta(D)]}) \geq D - \kappa$.

*Proof.* Assume by contradiction that there exists a path $\mathfrak{p}$ satisfying (1), (2) and (3). Let $a = \gamma(t_1)$ and $b = \gamma(t_1 + \ell)$. Modify $\mathfrak{p}$ by appending the segment $\gamma^{-1}|_{[b, \gamma(t_2)]}$. By Lemma 2.3, we have that the new path $\mathfrak{p}$ satisfies $\text{length}(p) \leq \chi'(\ell)$ for a linear function $\chi'$. For large enough $\ell$, and since $t_2 - t_1 \geq \ell$, this implies $\text{length}(\mathfrak{p}) \leq c \cdot \text{d}(a, b)$ for some constant $c$ only depending on $\chi'$ and $(\lambda, \kappa)$. By potentially increasing $\delta$ (by an amount that only depends on $(\lambda, \kappa)$) the new path $\mathfrak{p}$ still satisfies Condition (3). By Theorem 2.23, there exists $\mathfrak{t} \in (0, \frac{1}{2})$ and a middle recurrence $\mathfrak{m}_\mathfrak{t}$, both depending only on $M$, such that $\gamma$ is $\mathfrak{m}_\mathfrak{t}$–middle recurrent. Thus, $\mathfrak{p}$ intersects the $\mathfrak{m}_\mathfrak{t}(c)$–neighbourhood of the $\mathfrak{t}$–middle $\gamma_{\mathfrak{t}\cdot ab}$. Choose $D$ as $\mathfrak{m}_\mathfrak{t}(c) + \kappa + 1$. If there exists $\ell$ such that $\gamma|_{\mathfrak{t}\cdot ab} \subseteq \gamma|_{[t_1+\delta(D), t_1+\ell-\delta(D)]}$, this is a contradiction (3) and hence concludes the proof.

Lastly, we show how to choose such an $\ell$. Observe that the distance between $a$ and $\gamma|_{\mathfrak{t}\cdot[a,b]}$ has a lower bound that grows linearly in $\ell$, whereas the Hausdorff distance between $a$ and $\gamma|_{[t_1, t_1+\delta(D)]}$ has a uniform upper bound. A similar observation for $b$ and $\gamma|_{[t_1+\ell-\delta(D), t_1+\ell]}$ shows that for $\ell$ large enough $\gamma|_{\mathfrak{t}\cdot ab} \subseteq \gamma|_{[t_1+\delta(D), t_1+\ell-\delta(D)]}$. □

We conclude with another implication between Morse properties. We show that if a path is locally weakly Morse, then locally has a property akin to the middle recurrence for paths formed by quasi-geodesics.

**Lemma 2.25.** *Let $X$ be a geodesic metric space. Let $N, C, \varepsilon$ be constants, let $(\lambda_0, \kappa_0), (\lambda, \kappa)$ be a quasi-geodesic pair and let $M$ be a Morse gauge. There exists a constant $D \geq 0$ and a quasi-geodesic pair $(Q, q)$ such that if any path $\gamma$ is $R$–locally a $(Q, q, M(Q, q))$–weakly Morse $(\lambda, \kappa)$–quasi-geodesic for a constant $R \geq D$, then the following holds. If $u = \gamma(t)$ and $v = \gamma(s)$ with $|t - s| \leq R$, then every path from $u$ to $v$ which has length at most $CR$ and is composed of at most $N$ $(\lambda_0, \kappa_0)$–quasi-geodesics intersects the $\varepsilon R$–neighbourhood of every point of $\gamma|_{uv}$.*

*Proof.* Assume that for every $n$, there exists a $(\lambda, \kappa)$–quasi-geodesic $\gamma_n$ which is $(n, n, M(n))$–weakly Morse at scale $R_n \geq D_n = M(n, n)n$ but contains points $u_n, v_n$ at parametrized distance $R_n$ joined by a path composed of at most $N$ $(\lambda_0, \kappa_0)$–quasi-geodesics and of length at most $CR_n$ disjoint from the ball $B(m_n, \varepsilon R_n)$ for some $m_n \in \gamma_n|_{u_n v_n}$.

This means that in an asymptotic cone $\mathrm{Cone}_\omega(X, m_n, R_n)$, the points $A = (u_n)^\omega$ and $B = (v_n)^\omega$ can be connected by a path $\mathfrak{q}_\omega = \lim_\omega(\mathfrak{q}_n)$ composed of $N$ $\lambda_0$-bilipschitz arcs avoiding $m = (m_n)^\omega$. In particular, there exists a path $\mathfrak{p}$ from $A$ to $B$ which avoids $m$ and hence there exists a simple path from $A$ to $B$ which avoids $m$. Let $L$ be the set of geodesics in $X$ and let $L_\omega$ be the set of $\omega$-limits of those geodesics. By [11, Lemma 2.3] there exists a simple path $\tilde{\mathfrak{p}}$ from $A$ to $B$ which is a piecewise $L_\omega$ path and avoids $m$. Finally, by [11, Lemma 2.6 (2)] there exist constants $k$ and $\lambda$ such that $\tilde{\mathfrak{p}} = \lim^\omega \mathfrak{p}_n$ where each $\mathfrak{p}_n$ is $\lambda$–bilibschitz and a $k$–piecewise $L$ path. Now, for $n \geq \lambda$ we can use that $\gamma_n$ is $(n, n, M(n, n))$–weakly Morse to get that $\mathfrak{p}_n$ is in the $M(n, n)$–neighbourhood of $\gamma_n|_{u_n v_n}$. Since $R_n \geq D_n = M(n, n)n$, we have that $\lim^\omega(\gamma_n|_{u_n v_u})$ coincides with $\tilde{\mathfrak{p}}$, a contradiction that $\tilde{\mathfrak{p}}$ is disjoint from $m$. □

## 3. Globalization of the quasi-geodesic property

In this section, we prove that in a metric space with a bounded combing a path that is locally quasi-geodesic and Morse at a large enough scale is globally quasi-geodesic. We start with the following lemma about concatenations of quasi-geodesics. This is well-known and versions of this lemma have appeared in the literature (for instance [24]). We provide a proof for completeness.

**Lemma 3.1.** *Let $\gamma$ be a $(\lambda, \kappa)$–quasi-geodesics segment. Let $z_1, z_2 \in X$ and $u_1, u_2 \in \gamma$ be such that*

*(1) every point point $x \in \gamma$ between $u_1, u_2$ satisfies $\mathrm{d}(x, z_i) \geq \mathrm{d}(u_i, z_i)$.*

*Then, for all geodesics $\alpha_i$ connecting $u_i$ and $z_i$, the concatenations $\alpha_1 * \gamma|_{u_1 u_2}$ and $\gamma|_{u_1 u_2} * \alpha_2$ are $(2\lambda + 1, \kappa)$–quasi-geodesics. If, moreover, the following condition is satisfied*

*(2) $\mathrm{d}(u_i, z_i) \leq \theta \mathrm{d}(u_1, u_2)$ for some $\theta \in [0, 1/2)$;*

*then the concatenation $\alpha_1 * \gamma|_{u_1 u_2} * \alpha_2$ is a $(\lambda', \kappa)$–quasi-geodesic, where $\lambda' = \max\left(\frac{\lambda+1}{1-2\theta}, 2\lambda + 1\right)$.*

*Proof.* Let $\eta$ denote the concatenation $\alpha_1 * \gamma|_{u_1 u_2} * \alpha_2$, where $\alpha_i$ are geodesics joining $u_i$ and $z_i$, parameterized by their length, and let $\eta(s_i) = u_i$. By the triangular inequality, $\eta$ satisfies the quasi-geodesic upper inequality with constants $(\lambda, \kappa)$. In what follows, we focus on the lower inequality. Consider two points $x = \eta(s_x)$ and $y = \eta(s_y)$, with $s_x \leq s_y$. If $x, y$ are both contained in one of the three (quasi)-geodesics composing $\eta$, then the lower inequality is satisfied.

Consider now the case $x \in \alpha_1$ and $y \in \gamma|_{u_1 u_2}$ (the case $x \in \gamma|_{u_1 u_2}$ and $y \in \alpha_2$ is similar).

We have that $\mathrm{d}(x, y) \geq \mathrm{d}(x, u_1)$, otherwise $\mathrm{d}(z_1, y) < \mathrm{d}(z_1, u_1)$, contradicting the second assumption. It follows that $\mathrm{d}(x, y) \geq s_1 - s_x = \mathrm{d}(x, u_1)$, which implies that $\mathrm{d}(y, u_1) \leq 2\mathrm{d}(x, y)$. We can then write that

$$s_y - s_1 \leq \lambda \mathrm{d}(u_1, y) + \lambda\kappa \leq 2\lambda \mathrm{d}(x, y) + \lambda\kappa,$$

and therefore that

$$s_y - s_x = s_y - s_1 + s_1 - s_x \leq 2\lambda \mathrm{d}(x, y) + \lambda\kappa + \mathrm{d}(x, y) = (2\lambda + 1)\mathrm{d}(x, y) + \lambda\kappa.$$

This concludes the case of a single projection.

Assume now $x \in \alpha_1$, $y \in \alpha_2$. We have:

$$|s_x - s_y| \leq 2\theta \mathrm{d}(u_1, u_2) + |s_1 - s_2| \leq$$
$$\leq 2\theta \mathrm{d}(u_1, u_2) + \lambda \mathrm{d}(u_1, u_2) + \kappa = (2\theta + \lambda)\mathrm{d}(u_1, u_2) + \kappa.$$

As $\mathrm{d}(x, y) \geq (1 - 2\theta)\mathrm{d}(u_1, u_2)$, we conclude that

$$|s_x - s_y| \leq \frac{2\theta + \lambda}{1 - 2\theta}\mathrm{d}(x, y) + \lambda\kappa \leq \frac{\lambda + 1}{1 - 2\theta}\mathrm{d}(x, y) + \lambda\kappa.$$

□

We recall a basic fact about quasi-geodesics we are going to need in the following theorem. We include a proof to emphasize the linear dependence of the constants.

**Lemma 3.2.** *Let $\gamma_i$, $i = 1, 2$, be two $(\lambda, \kappa)$–quasi-geodesic segments with endpoints at distance $d$. Then for all $\mu \geq d$ there exists $\mu' = (1 + 2\lambda^2)\mu + \lambda\kappa + \kappa$ such that if $\gamma_1 \subseteq \mathcal{N}_\mu(\gamma_2)$ then $\gamma_2 \subseteq \mathcal{N}_{\mu'}(\gamma_1)$.*

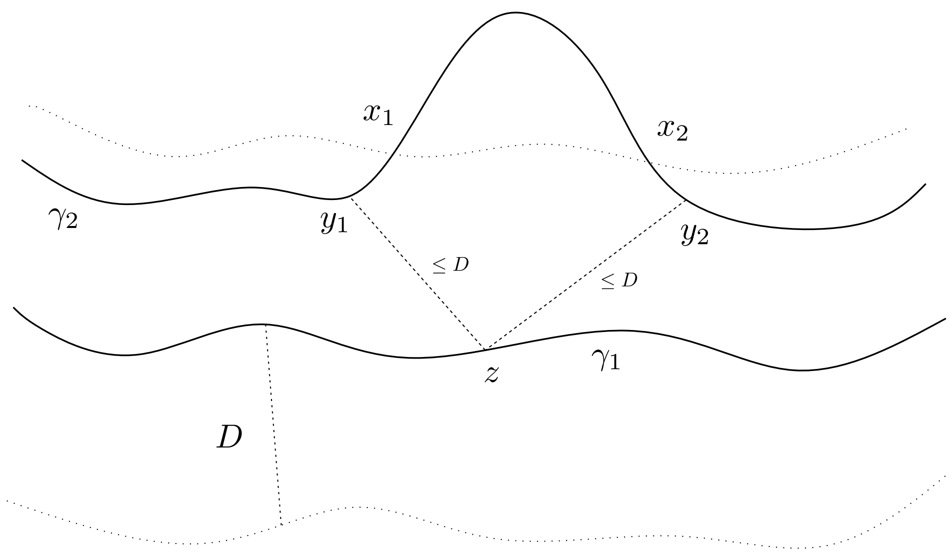


Figure 1. Proof of Lemma 3.2

*Proof.* Declare an order on $\gamma_2$ by ordering its endpoints and call them $a$ and $b$. We will bound the diameter of the connected components of $\gamma_2 \setminus \mathcal{N}_\mu(\gamma_1)$. Consider the longest such component, and let $x_1, x_2$ be its endpoints. Observe that there needs to be points $y_1$ before (or equal to) $x_1$, $y_2$ after (or equal to) $x_2$ and $z \in \gamma_1$ such that $\mathrm{d}(y_i, z) \leq \mu$. If not, $\gamma_1$ could be covered by the closed disjoint sets $\mathcal{N}_\mu(\gamma_2|_{ax_1}) \sqcup \mathcal{N}_\mu(\gamma_2|_{x_2 b})$, contradicting the assumption that quasi-geodesics are continuous. Thus, $\mathrm{d}(y_1, y_2) \leq 2\mu$, and hence $\mathrm{d}(m, \{y_1, y_2\}) \leq \lambda(2\lambda\mu + \kappa) + \kappa$ for all $m \in \gamma_2$ between $y_1, y_2$, yielding that $\gamma_2 \subseteq \mathcal{N}_{\mu'}(\gamma_1)$ where $\mu' = \lambda(2\lambda\mu + \kappa) + \kappa + \mu$. □

**Lemma 3.3.** *Let $X$ be a geodesic metric space with a bounded $(\lambda_0, \kappa_0)$–quasi-geodesic combing. For every $\lambda \geq 1$ and $\kappa \geq 0$, and for every $\mu \geq 0$ there exists $D \geq 0$ and $\mu' \geq 0$ such that every continuous path $\mathfrak{p}$ that is a $(\lambda, \kappa)$–quasi-geodesic*

*$(2\lambda_0+2,\kappa_0,\mu)$–weakly Morse at scale $D$ is contained in the $\mu'$–neighbourhood of the combing line between its endpoints.*

*Proof.* We can assume that $\mathfrak{p}$ is parameterized as $\mathfrak{p}\colon [0,a]\to X$ for some $a$. Let $k=\lceil a\rceil$ and let $\mathfrak{q}_i$ be the combing line between $\mathfrak{p}(0)$ and $\mathfrak{p}(i)$ for $i<k$, and $\mathfrak{q}_k$ be the combing line between $\mathfrak{p}(0)$ and $\mathfrak{p}(a)$.

Write $\mu'=\theta D$ and assume that $\mathfrak{p}$ is not in the $\theta D$–neighbourhood of $\mathfrak{q}_k$. By a (coarse) continuity argument there exists $\mathfrak{q}_i$ such that the corresponding subpath $\mathfrak{p}|_{[0,i]}$ is in the $\theta D+O(1)$–neighbourhood of $\mathfrak{q}_i$ but not in the $\theta D$–neighbourhood of $\mathfrak{q}_i$.

We fix such a $\mathfrak{q}_i$ and from now on we abuse notation and simply denote $\mathfrak{q}_i$ and $\mathfrak{p}|_{[0,i]}$ as $\mathfrak{q}$ and $\mathfrak{p}$, respectively. Let $\mathfrak{p}(s)$ be a point at distance at least $\theta D$ from $\mathfrak{q}$. Let $\rho\in(0,1/2)$ be another small constant to be determined. Let $t_1=\max(0,s-\rho D)$ and $t_2=\min(a,s+\rho D)$. We can assume that $a\geq\rho D$ and hence $D\geq 2\rho D\geq t_2-t_1\geq\rho D$.

For $i=1,2$, let $u_i\in\mathfrak{q}$ be a closest point to $\mathfrak{p}(t_i)$. Since $\mathrm{d}(u_i,\mathfrak{p}(t_i))\leq\theta D+O(1)$, we have that

$$\begin{aligned}\mathrm{d}(u_1,u_2)&\geq \mathrm{d}(\mathfrak{p}(t_1),\mathfrak{p}(t_2))-2\theta D-2\mathcal{O}(1)\geq\frac{(t_2-t_1)}{\lambda}-\kappa-2\theta D-2\mathcal{O}(1)\\&\geq\left(\frac{\rho}{\lambda}-2\theta\right)D-\kappa-2\mathcal{O}(1).\end{aligned}$$

Here we used that $\mathfrak{p}$ is $D$–locally a $(\lambda,\kappa)$–quasi-geodesic. Whence $\mathrm{d}(u_i,z_i)\leq\frac{1}{6}\mathrm{d}(u_1,u_2)$ if $\theta<\frac{\rho}{8\lambda}$ and $D$ is large enough. By Lemma 3.1, it follows that if $\alpha_i$ are geodesics connecting $\mathfrak{p}(t_i)$ to $u_i$, then the concatenation $\gamma=\alpha_1*\mathfrak{q}|_{u_1u_2}*\alpha_2$ is a $(\lambda_0',\kappa_0)$-quasi-geodesic, where $\lambda_0'=2\lambda_0+2$, joining two points on $\mathfrak{p}$ at parameterized distance less than $D$, implying that $\gamma$ is in the $\mu$–neighbourhood of $\mathfrak{p}|_{t_1t_2}$. By Lemma 3.2, we consequently have that $\mathfrak{p}|_{[t_1,t_2]}$ and in particular $\mathfrak{p}(s)$ is in the $r$–neighbourhood of $\gamma$ for some $r$ only depending on $\mu,\lambda_0,\kappa_0,\lambda$ and $\kappa$. For $\mu'=\theta D$ larger than $r$, $\mathfrak{p}(s)$ is not in the $r$ neighbourhood of $\alpha_1$ and $\alpha_2$ and hence not in the $r$–neighbourhood of $\mathfrak{q}$, a contradiction. □

**Remark 3.4.** If a combing line $\mathfrak{q}$ has the same endpoints as the quasi-geodesic $\gamma$ of Lemma 3.3, by Lemma 3.2 there exists $\mu''=\mu''(\lambda,\kappa,\lambda_0,\kappa_0,\mu)$ such that

$$d_{\mathrm{Haus}}(\gamma,\mathfrak{q})\leq\mu''.$$

**Lemma 3.5.** *Let $X$ be a geodesic metric space. Then for each quasi-geodesic pair $(\lambda,\kappa),(\lambda_0,\kappa_0)$ and constant $r$ there exists a scale $D$ and quasi-geodesic constants $(\lambda',\kappa')$ such that if $\mathfrak{p}$ is $D$–locally a $(\lambda,\kappa)$–quasi-geodesic contained in the $r$ neighbourhood of a $(\lambda_0,\kappa_0)$–quasi-geodesic $\gamma$, then $\mathfrak{p}$ is a $(\lambda',\kappa')$–quasi-geodesic.*

*Proof.* Since the statement for paths $\mathfrak{p}$ with unbounded domain follows directly from the statement for paths $\mathfrak{p}$ with bounded domain, we can and will assume that the domain of $\mathfrak{p}$ is bounded. By increasing $r$ a uniform amount (and possibly passing to a subsegment of $\gamma$) we can assume that the closest point projections of the endpoints of $\mathfrak{p}$ on $\gamma$ are exactly the endpoints of $\gamma$.

For each $i\geq 0$ where defined, let $x_i=\mathfrak{p}\left(i\frac{D}{2}\right)$ and let $y_i=\gamma(s_i)$ be a closest point projection of $x_i$ onto $\gamma$. Since $\mathfrak{p}$ is a $D$–local quasi-geodesic, the map $i\mapsto x_i$ satisfies the upper coarse Lipschitz inequality. We focus on the lower one. Since $\mathfrak{p}$ is a quasi-geodesic at scale $D$, $|s_{i+1}-s_i|\leq D/c-c$ for some constant $c$ depending on $r,(\lambda,\kappa)$ and $(\lambda_0,\kappa_0)$ but not $D$.

By the choice of parameterization, we can assume that $s_1 - s_0 \geq 0$. We will show inductively that for large enough $D$, $s_{i+1} - s_i \geq 0$ for all $i$ and hence $s_{i+1} - s_i \geq D/c - c$, which concludes the proof since it yields a linear lower bound on the lower coarse Lipschitz inequality for $\mathfrak{p}$.

Assume that $s_i - s_{i-1} \geq 0$. If $s_{i-1} \leq s_{i+1} \leq s_i$, then by Lemma 3.2, there exists $(i-i)\frac{D}{2} \leq t \leq i\frac{D}{2}$ such that $\mathrm{d}(\mathfrak{p}(t), \gamma(s_{i+1})) \leq r'$, where $r'$ depends on $r$ but not on $D$. Consequently, $\mathrm{d}(\mathfrak{p}(t), \mathfrak{p}((i+1)\frac{D}{2}))$ is bounded by a constant independent of $D$. For large enough $D$, this contradicts $\mathfrak{p}$ being a $(\lambda_0, \kappa_0)$–quasi-geodesic at scale $D$.

Similarly, if $s_{i+1} \leq s_{i-1} \leq s_i$ we get a contradiction by replacing the roles of $(i-1)$ and $(i+1)$. Therefore, we have to have $s_{i+1} > s_i$, which concludes the induction and hence the proof. □

Combining Lemmas 3.3 and 3.5, and Remark 3.4 we obtain the following.

**Theorem 3.6.** *Let $X$ be a geodesic metric space with a bounded $(\lambda_0, \kappa_0)$–quasi-geodesic combing. For every quasi-geodesic pair $(\lambda, \kappa)$ and $\mu \geq 0$, there exists a scale $D \geq 0$, and quasi-geodesic constants $(\lambda', \kappa')$ such that every path $\mathfrak{p}$ that $D$–locally is a $(\lambda, \kappa)$–quasi-geodesic that is also $(2\lambda_0 + 2, \kappa_0, \mu)$–weakly Morse, is a global $(\lambda', \kappa')$–quasi-geodesic.*

*Moreover, for every pair of points $u$ and $v$ in the image of $\mathfrak{p}$, the subpath $\mathfrak{p}|_{uv}$ and the combing line $\mathfrak{q}_{uv}$ are within Hausdorff distance at most $\mu'' = \mu''(\lambda, \kappa, \lambda_0, \kappa_0, \mu)$ of each other.*

## 4. Path systems

In order to establish that a certain geodesic is Morse, we will show that is has some contracting property with respect to a special set of paths, which in our case are, unsurprisingly, going to be related to combing lines. This line of reasoning was developed in [26].

**Definition 4.1** ([26, Definition 2.1])**.** *A* path system *$\mathcal{P}$ in $X$ is a collection of $(\kappa_0, \lambda_0)$-quasi-geodesics in $X$, for some quasi-geodesic constants $(\kappa_0, \lambda_0)$, such that:*

*(1) any subpath of a path in $\mathcal{P}$ is in $\mathcal{P}$,*
*(2) all pairs of points in $X$ can be connected by a path in $\mathcal{P}$.*

*Elements of $\mathcal{P}$ will be called* special paths*, and we denote a special path between $x, y$ by $\mathfrak{h}_{xy}$.*

**Definition 4.2** ([26, Definition 2.2])**.** *A subset $A \subseteq X$ will be called $\mathcal{P}$–contracting with constant $C$ if there exists a map $\pi_A \colon X \to A$ such that*

*(1) $\mathrm{d}(x, \pi_A(x)) \leq C$ for each $x \in A$,*
*(2) for each $x, y \in X$, if $\mathrm{d}(\pi_A(x), \pi_A(y)) \geq C$ , then for any special path $\mathfrak{h}_{xy}$ from $x$ to $y$ we have $\mathrm{d}(\mathfrak{h}_{xy}, \pi_A(x)) \leq C$, $\mathrm{d}(\mathfrak{h}_{xy}, \pi_A(y)) \leq C$.*

Note that a subset that is $\mathcal{P}$–contracting with constant $C$ is also $\mathcal{P}$–contracting with constant $C'$, for every $C' \geq C$.

**Theorem 4.3** ([26, Lemma 2.8 (1)])**.** *Let $\mathcal{P}$ be a path system in the metric space $X$. For every $C$ there exists $D$ so that if $A \subset X$ is $\mathcal{P}$–contracting with constant $C$, then any $(C, C)$–quasi-geodesic with endpoints in $A$ is contained in $N_D(A)$. In particular, if $\gamma$ is a quasi-geodesic which is $\mathcal{P}$–contracting with constant $C$, then $\gamma$ is $M$–Morse for some Morse gauge $M$ only depending on $C$, $\mathcal{P}$ and $X$.*

In what follows, we consider the path system given by all the combing lines of a quasi-consistent quasi-geodesic bicombing, their subpaths and their inverses. We use the notation $\mathfrak{q}_{xy}$ to denote the combing line from $x$ to $y$ and $\mathfrak{h}_{xy}$ to denote a special path from $x$ to $y$ (which can be a combing line, a subpath of a combing line, or an inverse of the former).

Let $\mathfrak{h}$ in $\mathcal{P}$ be a special path and let $\mathfrak{q}_{xy}$ be a combing line with endpoints on $\mathfrak{h}$. We say that $\mathfrak{q}_{xy}$ *goes in the same direction* as $\mathfrak{h}$ if there exists $x', y' \in X$ such that one of the following holds

- $\mathfrak{h}$ is a subsegment of $q_{x'y'}$ and there exists $s \leq t$ such that $x = \mathfrak{h}(s)$ and $y = \mathfrak{h}(t)$.
- $\mathfrak{h}$ is a subsegment of $q^{-1}_{x'y'}$ and there exists $s \geq t$ such that $x = \mathfrak{h}(s)$ and $y = \mathfrak{h}(t)$.

Quasi-consistency of the bi-combing implies that if $\mathfrak{q}_{xy}$ goes in the same direction as $\mathfrak{h}$, then it is contained in the $2\kappa_0$–neighbourhood of $\mathfrak{h}$.

The following Lemma extends [23, Lemma 7.1] to the case of spaces with bi-combings.

**Lemma 4.4** (Circumnavigation Lemma)**.** *Let $X$ be equipped with a bounded quasi-consistent $(\lambda_0, \kappa_0)$–quasi-geodesic bicombing. Let $x_1, \ldots, x_n$ be distinct points in $X$, and consider paths $\alpha_i$ between $x_{i-1}, x_i$ (indices taken mod $n$), special paths with respect to the combing path system. Suppose that there is a point $m \in \alpha_1$ so that the ball $B(m, R)$ is disjoint from $\alpha_i$ for all $i \neq 1$. Let $a$ and $b$ be the first respectively last point of $\alpha_1$ intersecting $B(m, R)$. There exists a constant $c$ depending only on $(\lambda_0, \kappa_0)$ and a path $\mathfrak{p}$ from $a$ to $b$ of length* $\mathrm{length}(p) \leq cn(R + \kappa_0)$ *that avoids the ball $B(m, R - 3(\kappa_0 + 1))$. Moreover, $\mathfrak{p}$ is the concatenation of at most $6n + 6$ special paths.*

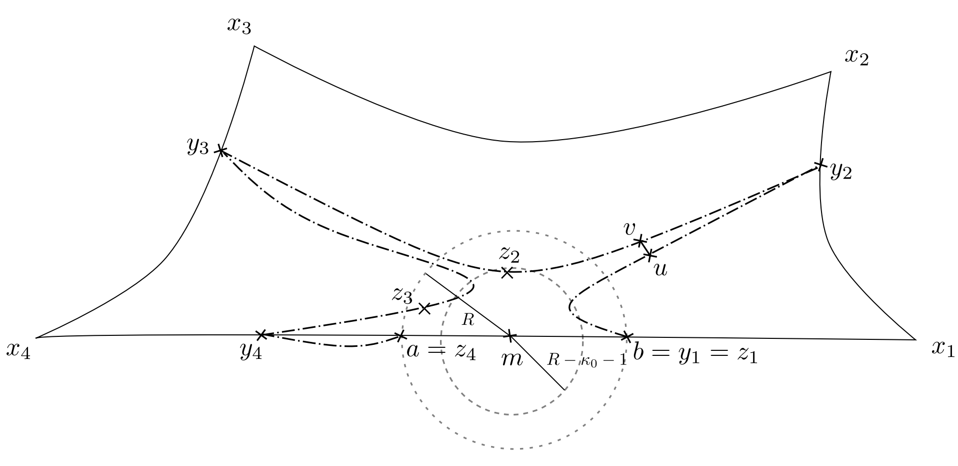


FIGURE 2. Proof of Lemma 4.4

*Proof.* Define $\alpha'_1 = \alpha_1|_{[b,x_2]}, \alpha'_{n+1} = \alpha_1|_{[x_1,a]}$ and $\alpha'_i = \alpha_i$ for all $2 \leq i \leq n$. Note that for all $1 \leq i \leq n+1$, $\alpha'_i$ is a subsegment of a combing line or the inverse of a combing line and $\mathrm{d}(\alpha'_i, m) \geq R$. Define

$$\alpha = \alpha'_1 * \alpha'_2 * \ldots * \alpha'_{n+1}.$$

Again, we have that $\mathrm{d}(\alpha, m) \geq R$. Let $I = [0, T]$ be the domain of $\alpha$.

Next, define $t_1 = 0$ and $y_1 = \alpha(t_1) = b$. For $i \geq 1$, inductively define $t_{i+1}$ to be the minimal index satisfying $t_i \leq t_{i+1} \leq T$ and

$$\mathrm{d}(\mathfrak{q}_{y_i\alpha(t_{i+1})}, m) \leq R - \kappa_0 - 1, \qquad \text{if } i \text{ is even,}$$
$$\mathrm{d}(\mathfrak{q}_{\alpha(t_{i+1})y_i}, m) \leq R - \kappa_0 - 1, \qquad \text{if } i \text{ is odd.}$$

Denote the path $\mathfrak{q}_{y_i\alpha(t_{i+1})}$ (respectively the path $\mathfrak{q}_{\alpha(t_{i+1})y_i}$) by $\mathfrak{q}_i$ if $i$ is even ($i$ is odd).

If no such index exists, define $t_{i+1} = T$. Define $y_{i+1} = \alpha(t_i)$.

Note that by boundedness of the combing and minimality of $t_{i+1}$, we have that $\mathrm{d}(\mathfrak{q}_{y_iy_{i+1}}, m) \geq R - 2\kappa_0 - 2$. Assume that $y_i, y_{i+1}$ and $y_{i+2}$ all lie on $\alpha'_j$ for some $j$. By the quasi-consistency of the bicombing, and because $\mathrm{d}(\alpha'_i, m) \geq R$ we know that at least one of $\mathfrak{q}_i$ and $\mathfrak{q}_{i+1}$ goes in the same direction as $\alpha'_j$, implying that it is contained in the $\kappa_0$–neighbourhood of $\alpha'_j$, which implies that at least one of $\mathfrak{q}_i, \mathfrak{q}_{i+1}$ does not intersect the $R - \kappa_0 - 1$ ball around $m$. Hence, unless $y_{i+2} = a$, $y_{i+2}$ cannot lie on $\alpha'_j$ if $y_i$ lies on $\alpha'_j$. Consequently, $y_{2n+2} = a$.

Next, for $1 \leq i \leq 2n+1$, let $z_i$ be a point on $\mathfrak{q}_i$ with

$$R - 2\kappa_0 - 2 \leq \mathrm{d}(z_i, m) \leq R.$$

Such a point always exists by construction. Further, assume that $z_1 = b$ and $z_{2n+1} = a$.

**Claim 1.** *There exists a constant $c$ only depending on $(\kappa_0, \lambda_0)$ such that for each $1 \leq i \leq n$ there exists a path $\mathfrak{p}_i$ from $z_i$ to $z_{i+1}$ of length $\mathrm{length}(\mathfrak{p}_i) \leq c(R + \kappa_0)$ and $\mathrm{d}(m, \mathfrak{p}_i) \geq R - 3\kappa_0 - 2$.*

Clearly, the claim concludes the proof.

*Proof of Claim 1* We prove the claim in the case where $i$ is even. The case where $i$ is even works analogously by changing the order of the endpoints on the defined paths $\mathfrak{q}$ and $\mathfrak{q}'$. First note that $\mathrm{d}(z_i, z_{i+1}) \leq \mathrm{d}(z_i, m) + \mathrm{d}(m, z_{i+1}) \leq 2R$. Denote $\mathfrak{q}_{z_iy_{i+1}}$ by $\mathfrak{q}$ and $\mathfrak{q}_{z_{i+1}y_{i+1}}$ by $\mathfrak{q}'$. Since $i$ is even, $\mathfrak{q}_i$ and $\mathfrak{q}$ (respectively $\mathfrak{q}_{i+1}$ and $\mathfrak{q}'$) go in the same direction. Hence by consistency,

$$\mathrm{d}(m, \mathfrak{q}) \geq \mathrm{d}(m, \mathfrak{q}_i) - \kappa_0 \geq R - 3\kappa_0 - 2.$$

Similarly, by consistency and reversibility, we have that

$$\mathrm{d}(m, \mathfrak{q}') \geq \mathrm{d}(m, \mathfrak{q}_{i+1}) - \kappa_0 \geq R - 3\kappa_0 - 2.$$

By boundedness, $\mathrm{d}(\mathfrak{q}(t), \mathfrak{q}'(t)) \leq 2\kappa_0 R + \kappa_0$ for all $t$. By the triangular inequality, there is $c_1 = c_1(\lambda_0, \kappa_0)$ such that if two points $u, v$ satisfy $\mathrm{d}(u, v) \leq 2\kappa_0 R + \kappa_0$ and $\mathrm{d}(m, \{u, v\}) \geq c_1(R + \kappa_0)$, then $\mathrm{d}(q_{uv}, m) \geq R - 3\kappa_0 - 2$. Moreover, given $c_1$ we can find $c_j = c_j(c_1, \lambda_0, \kappa_0)$, $j = 2, 3$, such that if $\gamma$ is a $(\lambda_0, \kappa_0)$–quasi-geodesic, then

$$c_3(R + \kappa_0) \geq \mathrm{d}(\gamma(0), \gamma(c_2(R + \kappa_0))) \geq c_1(R + \kappa_0) + R.$$

Now we construct the path $\mathfrak{p}_i$. If $c_2(R + \kappa_0)$ is not in the domain of $\mathfrak{q}$, then by 2.3 the path

$$\mathfrak{p}_i = \mathfrak{q} * (\mathfrak{q}')^{-1}$$

Satisfied the desired criteria. Otherwise, let $u = \mathfrak{q}(c_2(R + \kappa_0))$ and $v = \mathfrak{q}'(c_2(R + \kappa_0))$. Since $\mathrm{d}(\mathfrak{q}(0), m) \leq R$, the triangular inequality and the choice of the constants above yields $\mathrm{d}(\mathfrak{q}uv, m) \geq R$. In particular, the path

$$\mathfrak{p}_i = \mathfrak{q}|_{[z_i,u]} * \mathfrak{q}_{uv} * (\mathfrak{q}'|_{[z_{i+1},v]})^{-1}$$

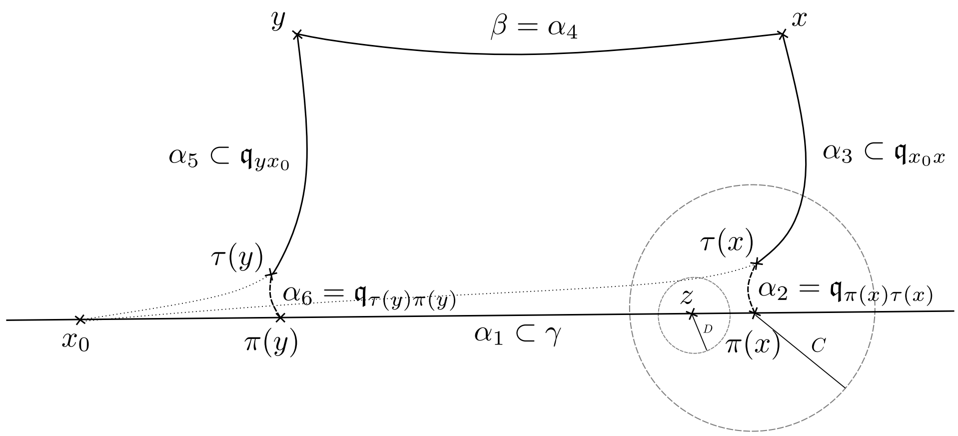


FIGURE 3. Using the circumnavigation Lemma.

satisfies $d(m, \mathfrak{p}_i) \geq R - 3\kappa_0 - 2$. Further, by Lemma 2.3 length$(\mathfrak{p}_i) \leq c(R + \kappa_0)$ for a constant $c$ depending only on $(\lambda_0, \kappa_0)$. ■

As there are at most $2n + 2$ points $z_i$, and since the paths $\mathfrak{p}_i$ constructed in the lemma are the concatenations of at most 3 segments, we get the results. □

We are now ready to prove the main result of the section.

**Theorem 4.5.** *Let $X$ be a geodesic metric space equipped with a bounded quasi-consistent $(\lambda_0, \kappa_0)$–bicombing. Then $X$ satisfies the MLTG property.*

*Proof.* Let $M$ be a Morse gauge and $(\lambda, \kappa)$ be quasi-geodesic constants. By Theorem 3.6, there exists $L' \geq 0$ and $(\lambda', \kappa')$ such that all $L'$–locally $M$–Morse $(\lambda, \kappa)$–quasi-geodesics $\gamma$ are global $(\lambda', \kappa')$–quasi-geodesics. It is left to prove that there exists a Morse gauge $M'$ and $L \geq 0$ such that any $(\lambda', \kappa')$–quasi-geodesic which is $L$–locally $M$–Morse is globally $M'$–Morse. In light of Lemma 3.3, it suffices to show this if $\gamma$ is a combing line.

Let $\mathcal{P}$ be the combing path system. By Theorem 4.3 it suffices to construct a projection $\pi : X \to \gamma$ and show that $\gamma$ is $\mathcal{P}$–contracting with constant $C$, where $C$ does not depend on $\gamma$.

**Choice of projection.** Let $x \in X$ and let $\tau(x)$ be the first point on $\mathfrak{q}_{xx_0}$ with $d(\tau(x), \gamma) \leq D$. Define $\pi(x)$ as a point on $\gamma$ with $d(\pi(x), \tau(x)) \leq D$.

**Claim 2.** *For large enough $D$, $\gamma$ is $\mathcal{P}$–contracting with constant $C$, where $C$ does not depend on $\gamma$.*

*Proof of Claim.* If $x \in \gamma$, then $\tau(x) = x$, so by the definition of $\pi$, we have that $d(x, \pi(x)) \leq D$. So for $C \geq D$, 4.2 (1) holds.

It remains to show 4.2 (2). Assume that 4.2 (2) does not hold. That is, there exists a path $\beta \in \mathcal{P}$ from $x$ to $y$ such that

$$d(\pi(x), \pi(y)) \geq C \tag{4.6}$$

and such that without loss of generality,

$$d(\beta, \pi(x)) \geq C. \tag{4.7}$$

Now consider the 6-gon $\alpha_1, \dots, \alpha_6$ with sides

$$\alpha_1 = \gamma|_{[\pi(y),\pi(x)]}, \qquad \alpha_2 = \mathfrak{q}_{\pi(x)\tau(x)},$$
$$\alpha_3 = \mathfrak{q}_{x_0x}|_{[\tau(x),x]}, \qquad \alpha_4 = \beta,$$
$$\alpha_5 = \mathfrak{q}_{yx_0}|_{[y,\tau(y)]}, \qquad \alpha_6 = \mathfrak{q}_{\tau(y)\pi(y)}.$$

Let $z$ be a point on $\alpha_1$ with $\mathrm{d}(z, \pi(x)) = C/2$. By the definition of $\pi$, we have that $\mathrm{d}(z, \alpha_3) \geq D$ and $\mathrm{d}(z, \alpha_5) \geq D$. If $C/4 \geq D$, then $\mathrm{d}(\alpha_4, z) \geq D$ by (4.7). Further, choosing $C$ large enough compared to $D$ we have that $\mathrm{diam}(\alpha_2) \leq C/4$ and $\mathrm{diam}(\alpha_6) \leq C/4$ and hence $\mathrm{d}(\alpha_2, z) \geq C/4 \geq D$ by the choice of $z$ and $\mathrm{d}(\alpha_6, z) \geq C/4 \geq D$ by (4.6).

This allows us to apply Lemma 4.4 with $m = z$ and $R = D$ to get $a, b$ on $\gamma$ and a path $\mathfrak{p}$ from $b$ to $a$ which does not intersect the $D - 3(\kappa_0 + 1)$–neighbourhood of $z$, and which is a concatenation of at most 42 special paths. Further, the length of $\mathfrak{p}$ is linearly bounded in $D$. For large enough $D$, we have that $D/2 \geq 3(\kappa_0 + 1)$ and hence $\mathfrak{p}$ does not intersect the $D/2$ neighbourhood of $z$. Lemma 2.25 shows that for large enough $D$ and $L$, such a path cannot exist. This concludes the proof. □

## 5. Linearity of divergence

A bi-product of Lemma 4.4 is that linearity of the divergence on a sequence diverging to infinity implies linearity of the divergence.

This type of result was previously known, under appropriate assumptions, for very few quasi-isometry invariants (the Dehn function and the growth function). Same as for the other two invariants, the above translates into a result of the type: a property known for one asymptotic cone propagates to all asymptotic cones.

**Theorem 5.1.** *Let $X$ be a geodesic metric space equipped with a bounded quasi-consistent $(\lambda_0, \kappa_0)$–bicombing. Assume that for every point $x \in X$ the ball $B(x, \kappa_0)$ intersects a bi-infinite $(\lambda_0, \kappa_0)$-quasi-geodesic from the bicombing (i.e. with $k_0$-tubular neighbourhood containing longer and longer bicombing lines). If there exists a sequence $n_k$ diverging to infinity such that on that sequence the divergence is bounded by a linear function then the divergence function is bounded by a linear function for every value.*

*Proof.* Since all the divergence functions are equivalent, we have a choice on the function to use for the hypothesis and the one to use for the conclusion. Thus, we assume that $\mathrm{Div}(n_k, \delta) \leq Cn_k$, for $\delta > 0$ smaller enough compared to the other constants, for instance $\delta < (10(\lambda_0 + \kappa_0 + 1))^{-2}$, and we aim to prove that $\mathrm{div}'(n, 2, \delta) \leq C'n$ for every $n$. Consider a pair of points $a, b$ at distance $2n$ and $m$ a mid-point on a geodesic joining $a$ and $b$, so that $\mathrm{d}(m, \{a, b\}) = n$. Without loss of generality, we can assume that $n$ is much larger than the combing constants $\lambda_0, \kappa_0$. First, if there is a combing line between $a, b$ that avoids the ball $B(m, \delta n)$, we are done. So, suppose it is not the case, and let $m'$ be a point on $\mathfrak{q}_{ab}$ closest to $m$. Observe that the ball of radius $2n\delta$ around $m'$ contains the ball of radius $n\delta$ around $m$. Our strategy will be to use Lemma 4.4 to find a path of controlled length avoiding $B(m', 2n\delta)$

Let $\gamma$ be a bi-infinite $(\lambda_0, \kappa_0)$-quasi-geodesic bicombing intersecting $B(a, \kappa_0)$ and $\gamma'$ be a bi-infinite $(\lambda_0, \kappa_0)$-quasi-geodesic bicombing intersecting $B(b, \kappa_0)$. Up to reparametrization, we can assume $\gamma(0) \in B(a, \kappa_0)$ and $\gamma'(0) \in B(b, \kappa_0)$. Consider the rays $\gamma|_{[0,\infty)}$ and $\gamma|_{[0,-\infty)}$. For $\delta > 0$ small enough and since $\mathrm{d}(\{a, b\}) = n$, we cannot have that both of those rays intersect $B(m', 2\delta n + 5\kappa_0)$, so let $\rho$ be one such ray that does not intersect $B(m', 2\delta n + 5\kappa_0)$. Likewise, select $\rho'$ to be a ray between $\gamma'|_{[0,\infty)}$ and $\gamma'|_{[0,-\infty)}$ that does not intersect $B(m', 2\delta n + 5\kappa_0)$. Let $k$ be such that $n_k \geq n$. By moving along $\rho$ and $\rho'$ we can find two points $x$ and $y$ at distance $n_k$. Since $\mathrm{Div}(n_k, \delta) \leq Cn_k$, we can find a path $\mathfrak{p}$ of length at most $Cn_k$ that connects $x, y$ while avoiding a ball of radius $n_k\delta$ around $m'$. Without loss of generality, $\mathfrak{p}$ is parametrized as $\mathfrak{p}\colon [0, T] \to X$ with $\mathfrak{p}(0) = x$ and $\mathfrak{p}(T) = y$. Inductively define parameters $s_i \in [0, T]$ by setting $s_0 = x$ and defining $s_i$ to be the highest value such that $\mathrm{d}(\mathfrak{p}(s_{i-1}), \mathfrak{p}(s_i)) = \theta n_k$ for a small $\theta > 0$ that we will describe shortly, or $s_i = T$ otherwise. Let $z_i = \mathfrak{p}(s_i)$ and consider a combing line $\mathfrak{q}_i = \mathfrak{q}_{z_i z_{i+1}}$. There are $N = N(\lambda_0, \kappa_0)$ and $\theta = \theta(\lambda_0, \kappa_0)$ so that if $n_k \geq N$, then

$$\mathrm{d}_{\mathrm{Haus}}(\mathfrak{q}_i, \mathfrak{p}([s_i, s_{i+1}])) \leq \frac{1}{2}\delta n_k - 3(\kappa_0 + 1).$$

Then the concatenation $\alpha = \mathfrak{q}_0 * \cdots * \mathfrak{q}_r$ is such that

(1) $r \leq \lceil \frac{C}{\theta} \rceil$,
(2) $\alpha$ connects $x$ and $y$,
(3) $\mathrm{d}_{\mathrm{Haus}}(\alpha, \mathfrak{p}) \leq \frac{1}{2}\delta n_k - 3(\kappa_0 + 1)$.

By choosing $n_k \geq 3n$, and since we assumed $n$ to be large compared to the bicombing constants, we get that $\alpha$ is disjoint from $B(m', 2n\delta + 3\kappa_0)$.

Let $\sigma$ be a combing line between $\rho(0)$ and $x$ if $\rho = \gamma|_{[0,\infty)}$ or a combing line between $x$ and $\rho(0)$ if $\rho = \gamma|_{[0,-\infty)}$. By the choice of $\rho$, the combing line $\sigma$ is disjoint from $B(m', 2n\delta + 3\kappa_0)$. Define analogously $\sigma'$ between $\rho'(0)$ and $y$ or between $y$ and $\rho'(0)$. Finally, consider combing lines between $\gamma(0), a$ and $b, \gamma'(0)$, and the combing line $\mathfrak{q}_{ab}$. The polygonal line $\alpha$ and the above combing lines form a polygon where all but one sides are disjoint from $B(m', 2n\delta + 3(\kappa_0 + 1))$, where the disjointedness is again obtained invoking that $n$ is large compared to $\lambda_0, \kappa_0$, and $\delta$ is small enough. Note that this is a $\left(\lceil \frac{C}{\theta} \rceil + 6\right)$-gon. We can then apply Lemma 4.4 to find two points $a', b' \in \mathfrak{q}_{ab}$ outside $B(m', 2\delta n)$ and a path $\eta$ connecting them and and avoiding $B(m', 2\delta n)$. Thus, we can find a path connecting $a, b$ and avoiding $B(m', 2\delta n)$ of length bounded above by $\ell(\eta) + \ell(\mathfrak{q}_{ab}) \leq c\left(\lceil \frac{C}{\theta} \rceil + 6\right)(2n\delta + 3(k_0 + 1)) + k_1 n + k_2 \leq C'n$, for some $C' = C'(\lambda_0, \kappa_0, \delta)$, where $k_1$ and $k_2$ are coming from Lemma 2.3 and depend only on $\lambda_0, \kappa_0$. As argued above, such a path will avoid $B(m, \delta n)$. □

## 6. Bicombings and weak Morse-local-to-global

The main goal of this section is to prove Theorem 6.6, namely showing that a metric space equipped with a bounded combing needs to satisfy the weak Morse local-to-global property.

In the proof of the theorem we use the following technical lemmas.

### 6.1. **Exit point and technical lemmas.**

**Lemma 6.1.** *Let $(\lambda, \kappa)$ be quasi-geodesic constants. There exists a linear function $\delta$ such that the following holds. Let $\gamma_1 : [0, T_1] \to X, \gamma_2 : [0, T_2]$ be $(\lambda, \kappa)$-quasi-geodesics such that $\gamma_2 \subseteq \mathcal{N}_D(\gamma_1)$. Let $x_i = \gamma_i(t_i), y_i = \gamma_i(s_i)$ be points such that $\mathrm{d}(x_1, x_2) \leq D$ and $\mathrm{d}(y_1, y_2) \leq D$. Assume that $t_1 \leq s_1$ and $t_2 \leq s_2$. Then $\mathrm{d}(\gamma_2|_{[t_2,s_2]}, \gamma_1(t)) > D$ for all $t \in [0, T_1] - [t_1 - \delta(D), s_1 + \delta(D)]$.*

*Proof.* We start by proving that $\mathrm{d}(\gamma_2|_{[t_2,s_2]}, \gamma_1(t)) > D$ for $t \in [0, t_1 - \delta(D)]$. Let $k_2$ be the largest parameter such that there exists $t_0 \leq t_1$ with $\mathrm{d}(\gamma_2(k_2), \gamma_1(t_0)) \leq D$.

Let $z_2 = \gamma_2(k_2)$. By continuity, there exists $k_1 \geq t_1$ such that $\mathrm{d}(z_2, \gamma_1(k_1)) \leq D$. In particular, $\mathrm{d}(\gamma_1(t_0), \gamma_1(k_1)) \leq 2D$, and hence $k_1 - t_0 \leq 2\lambda D + \kappa$. Using that, $t_0 \leq t_1 \leq k_1$ the fact that $\gamma_1$ and $\gamma_2$ are $(\lambda, \kappa)$–quasi-geodesics and the triangle inequality, we get that $k_2 - t_2$ are linearly bounded from above and hence there exists a linear function $\epsilon$ such that

$$\mathrm{d}(\gamma_2(s), x_2) \leq \epsilon(D),$$

for all $t_2 \leq s \leq k_2$. If $t < t_1 - \lambda(\epsilon(D) + 2D) + \kappa$. Then $\mathrm{d}(\gamma_1(t), x_1) \geq \epsilon(D) + 2D$, so $\mathrm{d}(\gamma_1(t), x_2) \geq \epsilon(D) + D$, and so we conclude that $\mathrm{d}(\gamma_1(t), \gamma_2|_{[t_2,s_2]}) \geq D$. The proof for $t \in [t_2 + \delta(D), T]$ is analogous. □

Given two quasi-geodesics starting at the same point, but ending far away from each other, we want to define the point at which they "exit their $D$–neighbourhoods for sufficently" long. This is made precise in the following definition of an exit point. The following results then describe the properties of those exit points.

**Definition 6.2.** *Let $\gamma : [0, T] \to X$ and $\eta : [0, T'] \to X$ be $(\lambda, \kappa)$–quasi-geodesics starting at a point $x = \gamma(0) = \eta(0)$. We say that a point $\gamma(t)$ on $\eta$ is a $(D, \ell)$–exit point of $(\eta, \gamma)$ if there exists a constant $t_e \in [0, T]$ with $\mathrm{d}(y, \gamma(t_e)) \leq D$ and such that*

$$\mathrm{d}(\eta|_{[t,T']}, \gamma|_{[0,t_e+\ell]}) \geq D.$$

*We call a minimal such $t_e$ the exit-moment.*

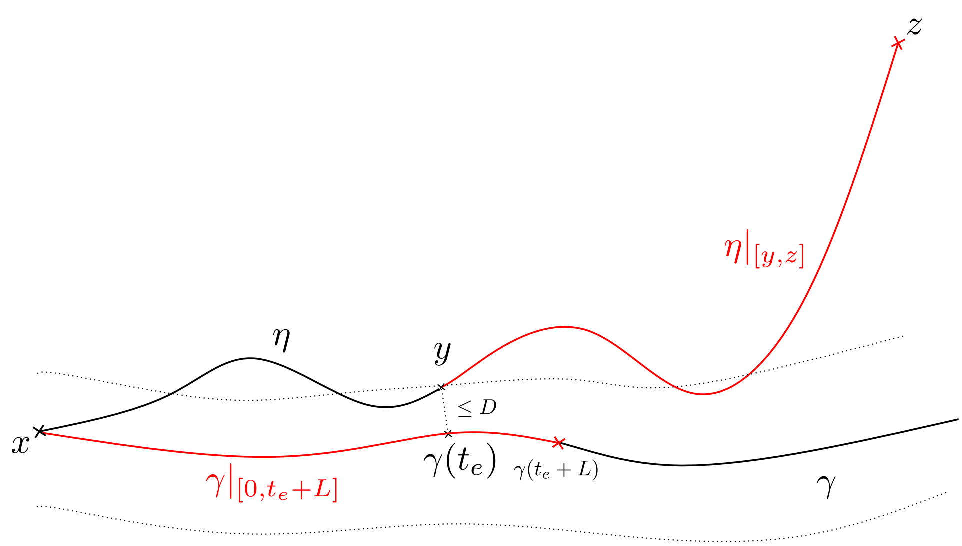


FIGURE 4. Definition of $(D, \ell)$–exit point.

Observe that if $\mathrm{d}(\eta(T'), \gamma) > D$, then, by continuity, a $(D, \ell)$–exit point exists, for example the last point on $\eta$ in the $D$–neighbourhood of $\gamma$.

The following lemma states that if $t_e$ is the minimal exit moment, then $\gamma|_[0, t_e]$ stays close to $\eta$.

**Lemma 6.3.** *Using the notation of Definition 6.2. If* $y = \eta(t)$ *is a* $(D, \ell)$*–exit point of* $(\eta, \gamma)$ *that minimizes the exit moment* $t_e$*, then for every* $s$ *with* $0 \le s \le t_e$ *there exists* $s'$ *with* $s \le s' \le s + \ell$ *such that* $\mathrm{d}(\gamma(s'), \eta) \le D$

*Proof.* Let $t' \in [0, T']$ be minimal such that $\mathrm{d}(\eta|_{[t',T']}, \gamma|_{[0,s+\ell]}) \ge D$. Such a point exists since $\eta(t)$ is a $(D, \ell)$–exit point with $t_e \ge s$. By continuity of $\eta$, $\mathrm{d}(\eta(t'), \gamma|_{[0,s+\ell]}) = D$. Let $0 \le u \le s + \ell$ be minimal such that $\mathrm{d}(\eta(t'), \gamma(u)) = D$. If $u < s$, then $\eta(t')$ is a $(D, \ell)$–exit point with exit moment $u < t_e$, contradicting the minimality of $t_e$. Hence $s \le u \le s + \ell$, implying that the lemma holds, since for example $s' = u$ works. □

**Corollary 6.4.** *Using the notation of Definition 6.2 and fixing quasi-geodesic constants* $(\lambda, \kappa)$*. There exists a linear function* $\nu$ *such that the following holds. If* $\eta(t)$ *is a* $(D, \ell)$*–exit point of* $(\eta, \gamma)$ *(which are both* $(\lambda, \kappa)$*–quasi-geodesics) that minimizes the exit moment* $t_e$*, then*

$$d_{\mathrm{Haus}}(\gamma([0, t_e]), \eta|_{[0,t]}) \le \nu(\ell + D).$$

*Proof.* By Lemma 6.3, for every $s \in [0, t_e]$ we have that $\mathrm{d}(\gamma(s), \eta) \le D + \lambda\ell + \kappa$. Lemma 3.2 concludes the proof. □

The following lemma shows that if a point on $\gamma$ which is before the minimal exit moment and a point on $\eta$ are close, then the start of $\gamma$ stays far away from the tail of $\eta$.

**Lemma 6.5.** *Using the notation of Definition 6.2 and fixing quasi-geodesic constants* $(\lambda, \kappa)$*. There exists a linear function* $\epsilon$ *such that the following holds. If* $\eta(t)$ *is a* $(D, \ell)$*–exit point of* $(\eta, \gamma)$ *(which are both* $(\lambda, \kappa)$*–quasi-geodesics) that minimizes the exit moment* $t_e$*, and* $0 \le s \le t_e$ *is a constant such that* $\mathrm{d}(\gamma(s), \eta(t')) \le D$ *for some* $t' \in [0, t]$*, then*

$$\mathrm{d}(\gamma|_{[0,s-\epsilon(D+\ell)]}, \eta|_{[t',T']}) \ge D$$

*Proof.* By Corollary 6.4 we have $d_{\mathrm{Haus}}(\gamma([0, t_e]), \eta|_{xy}) \le \nu(D + \ell)$. Let $D' = \nu(D + \ell) + D$. By Lemma 6.1, with the pair $\{\gamma(s), \eta(t')\}$ having the role of $\{x_i\}$ in the Lemma and $\{\gamma(t_e), \eta(t)\}$ the role of $\{y_i\}$, there exists $\delta$ such that $\mathrm{d}(\eta|_{[t',t]}, \gamma|_{[0,s-\delta(D')]}) \ge D'$. Since $D' \ge D$, and since the definition of exit point allows us to estimate $\mathrm{d}(\eta|_{[t,T']}, \gamma([0, s - \delta(D')]))$, the result follows. □

We are now ready to prove Theorem 6.6, which is the main theorem of this section.

**Theorem 6.6.** *Let* $X$ *be a geodesic metric space equipped with bounded* $(\lambda_0, \kappa_0)$*–quasi-geodesic combing. Let* $(\lambda, \kappa)$ *and* $(Q, q)$ *be quasi-geodesic pairs, and let* $M$ *be a Morse gauge. There exist constants* $L, N$ *such that the following holds. Let* $\gamma$ *be a* $(\lambda, \kappa)$*–quasi-geodesic that is* $M$*–Morse at scale* $L$*. Then* $\gamma$ *is* $(Q, q, N)$*–weakly Morse.*

*Proof.* In this proof we can and will assume that $\lambda \ge \lambda_0$ and $\kappa \ge \kappa_0$.

**Outline of the proof:** We will show that for large enough constants $N_1, N_2$ the following holds. If $\gamma$ is an $L$-locally $M$–Morse $(\lambda, \kappa)$–quasi-geodesic, then every $(Q, q)$-quasi-geodesic with endpoints in the closed $N_1$-neighbourhood of $\gamma$ stays in the $N_2$-neighbourhood of $\gamma$.

We do so by contradiction, assuming that the property does not hold and then finding a path $\mathfrak{p}$ with endpoints on $\gamma$ which contradicts Lemma 2.24.

We define $L, \ell, D$ as follows.

- $\delta_1, \sigma_1, \chi_1$ are the linear function from the proof of Claim 3
- $\delta, \sigma$ are the linear functions defined directly before Claim 5.
- $\chi$ is the function from Claim 6
- $\delta' \geq \delta_1, \delta, \sigma' \geq \sigma_1, \sigma, \chi' \geq \chi_1, \chi$ are linear functions.
- $D, \ell$ are the constants from Lemma 2.24 applied to $\delta', \sigma', \chi'$.
- $L = \sigma(\ell)$.

The constants $N_1$ and $N_2$ have to be large compared to the other constants (and $N_2$ has to be large compared to $N_1$). Throughout the proof, we outline the inequalities that $N_1$ and $N_2$ have to satisfy.

Let $\gamma$ be a $(\lambda, \kappa)$–quasi-geodesic which is $L$–locally $M$–Morse. Assume that there exist $(Q, q)$–quasi-geodesics with endpoints in the $N_1$–neighbourhood of $\gamma$ which do not stay in the $N_2$–neighbourhood of $\gamma$. Let $\eta$ be the shortest (in terms of length of its domain) such $(Q, q)$-quasi-geodesic. Observe the following; since $\eta$ is the shortest such quasi-geodesic, we have $\mathrm{d}(\eta, \gamma) \geq N_1$.

Let $z, z'$ be the endpoints of $\eta$ and let $x$ and $x'$ be the closest points on $\gamma$ to $z$ and $z'$ respectively. We may assume that $\gamma : [0, T] \to X$ and $\eta : [0, T'] \to X$ with $\gamma(0) = x, \gamma(T) = x', \eta(0) = z$ and $\eta(T') = z'$.

Let $n(\eta) = \lfloor T/(2\ell) \rfloor$. For each $0 \leq i \leq n(\eta)$, define

$$z_i = \eta(iT'/n(\eta)).$$

Fix a geodesic $[z', x']$ between $z'$ and $x'$. Define $n = n(\eta) + \lfloor \mathrm{length}([z', x']) - D \rfloor$. For each $n(\eta) + 1 \leq i \leq n$ define

$$z_i = [z', x'](i - n(\eta)).$$

Further for all $0 \leq i \leq n$, let $\mathfrak{q}_i : [0, T_i]$ be the combing line from $x$ to $z_i$. Let $y_i = \mathfrak{q}_i(s_i)$ be a $(D, \ell)$–exit point with minimal exit moment $t_i$ and let $x_i = \gamma(t_i)$.

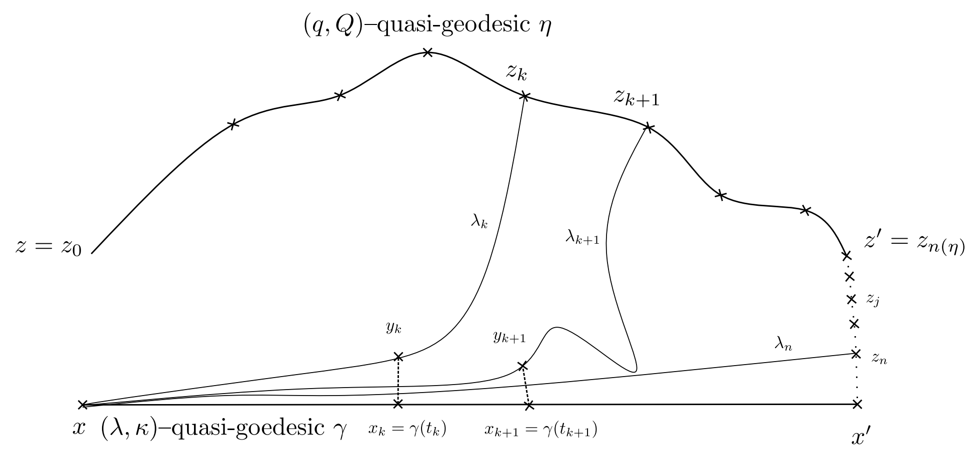


FIGURE 5. Setup for proof of Theorem 6.6

**Claim 3.** *There exists a constant $c_1$ only depending on $N_1, \kappa, \lambda, \ell, D$ (but not $N_2$) such that $t_0 \leq c_1$ and $t_n \geq T - c_1$.*

*Proof of Claim 3:* We have that $\mathrm{d}(x, z_0) = N_1$. Thus $\mathrm{Dlength}(\mathfrak{q}_0) \leq \lambda N_1 + \lambda\kappa \leq 2\lambda N_1$ and hence $\mathrm{d}(x, \mathfrak{q}_0(t)) \leq \lambda 2\lambda N_1 + \kappa \leq 3\lambda^2 N_1$ for all $t$. Consequently, $\mathrm{d}(\gamma(t_0), x) \leq D + 3\lambda^2 N_1 \leq 4\lambda^2 N_1$, which implies that $t_0 \leq 4\lambda^3 N_1 + \lambda\kappa = c_1$.

used $N_1 \geq D$

$\mathrm{Dlength}(\mathfrak{q}_n)$ is the domain length of $\mathfrak{q}_n$, we used $N_1 \geq \lambda\kappa$

We next show that $T - t_n \leq \sigma_1(\ell)$ by arguing that if it was not the case then we could construct a path $\mathfrak{p}$ contradicting Lemma 2.24. Here $\sigma_1$ denotes a linear function which we detrmine later. Let $\mathfrak{q}_{n+1}$ be the combing line from $x$ to $x'$. By boundedness, the Hausdorff distance between $\mathfrak{q}_n$ and $\mathfrak{q}_{n+1}$ is at most $\lambda_0(D+1)+\kappa_0$. By Remark 3.4, there is $\mu$ depending only on the above constants such that the Hausdorff distance between $\gamma$ and $\mathfrak{q}_{n+1}$ is at most $\mu$. Consequently, there exists a point $y$ on $\mathfrak{q}_n$ with $\mathrm{d}(\gamma(t_n + \sigma_1(\ell)), y) \leq \mu + \kappa_0(D + 1) + \kappa_0$. If we define $\sigma_1(\ell) = \ell + \delta(\mu + \kappa_0(\ell + 1) + \kappa_0)$, where $\delta$ is the function from Lemma 6.1, then Lemma 6.1, implies that $y \in \mathfrak{q}_n|_{y_n z_n}$. Consider the path $\mathfrak{p} = [x_n, y_n] * \mathfrak{q}_n|_{y_n y} * [y, \gamma(t_n+\ell)]$. By Lemma 2.3 we have that $\mathrm{length}(\mathfrak{p}) \leq \chi_1(\ell)$ for a linear function $\chi_1$ only depending on $\kappa_0, \lambda_0, \kappa, \lambda$ and $\mu$. As the distance between $x_n, y_n$ and $y, \gamma(t_n+\ell)$ is uniformly bounded in terms of $D$ and the above constants and $\mathrm{d}(\gamma, \mathfrak{q}_n|_{y_n y}) \geq D$, there is a linear function $\delta_1$ such that $\mathfrak{p}$ does not intersect the $D$–neighbourhood of $\gamma|_{[t_n+\delta_1(D), t_n+\ell-\delta_1(D)]}$. The choice of $D$ and $\ell$ show that this is a contradiction to Lemma 2.24. ∎

Since $\eta$ is a $(Q, q)$–quasi-geodesic, requiring that $N_2$ is large compared to $N_1$ (and hence large compared to $\kappa$, $\lambda$, $\ell$ and $(Q, q)$ we can guarantee that $n(\eta)$ is large, or in other words, that $2c_1/n$ and $(n - n(\eta))/n(\eta)$ are as small as we like, in particular, we can assume that

$$2c_1/n < \ell/3, \tag{6.7}$$

$$n \leq \frac{3n(\eta)}{2}. \tag{6.8}$$

**Claim 4.** *There exists $0 \leq i \leq n-1$ such that $t_{i+1} - t_i > \ell$.*

*Proof of Claim 4* Claim 3 implies that $\sum_{i=0}^{n-1} t_{i+1} - t_i \geq T - 2c_1$. Thus, there exists $0 \leq i \leq n-1$ such that $t_{i+1} - t_i \geq T/n - 2c_1/n$. By (6.7) and (6.8),

$$\frac{T}{n} - \frac{2c_1}{n} > \frac{2T}{3n(\eta)} - \frac{\ell}{3} \geq \frac{4\ell}{3} - \frac{\ell}{3} \geq \ell.$$

∎

We now want to bound $\mathrm{d}(z_i, z_{i+1})$. For $i \geq n(\eta)$ we have that $\mathrm{d}(z_i, z_{i+1}) = 1$, so be proceed to bound $\mathrm{d}(z_i, z_{i+1})$ for $i < n(\eta)$. To do this, we first want to bound $T'$, i.e. the length of the domain of $\eta\colon [0, T'] \to X$. Observe that $T \leq 2(n(\eta)+1)\ell$ and hence $\mathrm{d}(x, x') \leq 2(n(\eta)+1)\ell\lambda + \kappa$. Therefore by the triangle inequality,

$$\mathrm{d}(z_0, z_n) \leq 2(n(\eta)+1)\ell\lambda + \kappa + 2N_1.$$

Since $\eta$ is a $(Q, q)$–quasi-geodesic we have that

$$T' \leq Q(2(n(\eta)+1)\ell\lambda + \kappa + 2N_1) + Qq.$$

Recall that for $N_2$ large enough, $N_1/n(\eta)$ can be as small as we want. Hence

$$\begin{aligned} \frac{T'}{n(\eta)} &\leq Q(2\ell\lambda) + \frac{Q(2\ell\lambda + \kappa + 2N_1) + q}{n(\eta)} \\ &\leq Q(2\ell\lambda) + 1. \end{aligned}$$

Lastly we can see that

$$\text{(6.9)}\quad \mathrm{d}(z_i, z_{i+1}) = d\left(\eta\left(i\frac{T'}{n(\eta)}\right), \eta\left((i+1)\frac{T'}{n(\eta)}\right)\right) \le Q(Q(2\ell\lambda)+1)+q = C,$$

which gives the desired bound on $\mathrm{d}(z_i, z_{i+1})$.

Let $0 \le i \le n-1$ be such that $t_{i+1} - t_i > \ell$. Let $\sigma(x) = \epsilon(2x) + x$ be a linear function, where $\epsilon$ is the linear function from Lemma 6.5 applied to quasi-geodesic constants $(\lambda, \kappa)$. Let $\delta$ be the linear function from Lemma 6.1 applied to $(\lambda, \kappa)$-quasi-geodesics. Define $\gamma_R = \gamma|_{[t_i+\delta(D), t_i+\ell-\delta(D)]}$.

Recall that $\mathfrak{q}_i(s_i)$ is the $(D,\ell)$–exit point for $(\mathfrak{q}_i, \gamma)$. Hence we have that $\mathrm{d}(\mathfrak{q}_i(s_i), \gamma(t_i)) \le D$ and the following two paths have distance at least $D$ from $\gamma_R$

I) $\mathfrak{q}_i|_{[s_i,T_i]}$,
II) $[\gamma(t_i), \mathfrak{q}_i(s_i)]$.

Where for II) we used Lemma 6.1. We now want to find point $\gamma(t)$ and $\mathfrak{q}_{i+1}(s)$ with similar properties and where in addition $t - t_i$ is bounded from above and below in terms of $\ell$.

**Claim 5.** *There exists* $t' \in [0, T]$ *and* $s' \in [0, T_{i+1}]$ *such that* $\ell \le t' - t_i \le \sigma(\ell)$ *and* $\mathrm{d}(\gamma(t'), \mathfrak{q}_{i+1}(s')) \le D$. *Moreover, we can choose* $t'$ *and* $s'$ *such that the following paths have distance at least D from* $\gamma_R$

*III)* $\mathfrak{q}_{i+1}|_{s',T_{i+1}}$,
*IV)* $[\gamma(t'), \mathfrak{q}_{i+1}(s')]$.

*Proof of Claim 5.*

**Case 1:** $t_{i+1} - t_i \le \sigma(\ell)$**.** In this case, choose $t' = t_{i+1}$ and $s' = s_{i+1}$. Property III) and $\mathrm{d}(\gamma(t'), \mathfrak{q}_{i+1}(s')) \le D$ follow from $\mathfrak{q}_{i+1}(s')$ being a $(D,\ell)$–exit point. Since $t_{i+1} \ge t_i + \ell$, Property IV) follows from Lemma 6.1.

**Case 2:** $t_{i+1} - t_i > \sigma(\ell)$. In this case, $t_i + \sigma(\ell) - \ell \le t' \le t_i + \sigma(\ell)$ be such that $\mathrm{d}(\gamma(t'), \mathfrak{q}_{i+1}) \le D$. Choose $s'$ such that $\mathrm{d}(\mathfrak{q}_{i+1}(s'), \gamma(t')) \le D$. Again IV) follows from Lemma 6.1. Property III) follows from Lemma 6.5 and the choice of $\sigma$. ■

Define $a = \gamma(t_i), b = \gamma(t'), c = \mathfrak{q}_i(s_i)$ and $d = \mathfrak{q}_{i+1}(s')$. Now the goal is to find a path of controlled length that connects $c$ with $d$ and avoids III) follows from Lemma$\gamma_R$. The general idea is to "flow-up" the combing lines to get far away from $\gamma_R$ and then "jump" from one combing line to the other using the boundedness assumption to guarantee that the two combing lines are sufficiently close to each other.

**Defining the path $\mathfrak{p}$.** Observe that $\mathrm{d}(a, u) \le \lambda\sigma(\ell) + \kappa$ for all $u \in \gamma|_{ab}$. Define $\rho(\ell) = 2D + \lambda\sigma(\ell) + \kappa$. By the triangular inequality, $\mathrm{d}(c, d) \le \rho(\ell)$ and $\mathrm{d}(c, u) \le \rho(\ell)$ for all $u \in \gamma_R$. Recall that $c = \mathfrak{q}_i(s_i), d = \mathfrak{q}_{i+1}(s')$ and that $C$ is a bound on $\mathrm{d}(z_i, z_{i+1})$ linearly depending on $\ell$. Define

$$\text{(6.10)}\qquad \tau(\ell) = \lambda(\kappa C + 2\kappa + D + \rho(\ell)).$$

For large enough $N_1$, the domain length of $\mathfrak{q}_i$ is at least $s_i + \tau(\ell)$ and hence

$$\text{(6.11)}\qquad \mathrm{d}(\mathfrak{q}_i(\tau(\ell) + s_i), \mathfrak{q}_i(s_i)) \ge \kappa C + \kappa + D + \rho(\ell).$$

By boundedness, there exist $s$ such that

$$\text{(6.12)}\qquad \mathrm{d}(\mathfrak{q}_i(s_i), \mathfrak{q}_{i+1}(s)) \le \kappa C + \kappa.$$

Consider the following path

$$\text{(6.13)}\qquad \mathfrak{p} = [a, c] * \mathfrak{q}_i|_{[s_i, s_i+\tau(\ell)]} * [\mathfrak{q}_i(s_i + \tau(\ell)), \mathfrak{q}_{i+1}(s)] * \mathfrak{q}_{i+1}|_{[s,s']} * [d, b].$$

**Claim 6.** *The path $\mathfrak{p}$ satisfies the following two properties:*

*(1)* $\mathrm{d}(\mathfrak{p}, \gamma_R) \geq D$
*(2)* $\mathrm{length}(\mathfrak{p}) \leq \chi(\ell)$*, where* $\chi$ *is a function which is linear in* $\ell$ *and depends on* $(Q, q)$*.*

*Proof of Claim 6.* (1): By the triangle inequality, (6.11) and (6.12), we have that $\mathrm{d}([\mathfrak{q}_i(s_i + \tau(\ell)), \mathfrak{q}_{i+1}(s)], \gamma_R) \geq D$. Hence $\mathrm{d}(\mathfrak{p}, \gamma_R) \geq D$ follows from I), II), III) and IV).

(2): Since $\mathfrak{p}$ is a concatenation of five $(\lambda, \kappa)$–quasi-geodesics, by Lemma 2.3, it suffices to show that for each of them the distance of their endpoints is linearly bounded in $\ell$. For $[a, c], [b, d]$ this follows from II) and IV). For $\mathfrak{q}_i|_{[s_i, s_i+\tau(\ell)]}$ it follows since $\tau(\ell)$ is linear in $\ell$. For $[\mathfrak{q}_i(s_i + \tau(\ell)), \mathfrak{q}_{i+1}(s)]$ it follows from (6.12). Lastly, for $\mathfrak{q}_{i+1}|_{[s,s']}$ it follows from the triangle inequality, because both $\mathrm{d}(a, b)$ and all of the other four subsegments had endpoints at linearly bounded distance. Thus there exists a linear function $\chi$ such that $\mathrm{length}(p) \leq \chi(\ell)$. Not that $\tau$ depends on $(Q, q)$ and hence so does $\chi$ but this dependece is allowed. ■.

**Concluding the proof.** We now have a path $\mathfrak{p}$, which by Lemma 2.24 cannot exist, a contradiction. Hence, all $(Q, q)$–quasi-geodesic $\eta$ with endpoints in the $N_1$–neighbourhood of $\gamma$ stay in the $N_2$–neighbourhood. Implying any $(\lambda, \kappa)$–quasi-geodesic which is $L$–locally $M$–Morse is $(N_2, Q, q)$–weakly Morse. The scale $L$ depends on $(Q, q)$. □

## 7. Weak Morse local-to-global and sigma-compactness

The goal of this section is to show that a metric space with the weak Morse MLTG property needs to have either the (ordinary) MLTG property or have a non-$\sigma$ compact Morse boundary. We remind that a topological space is *$\sigma$–compact* if it can be written as a countable union of compact sets. Let us fix some notation. Given functions $M, N \colon X \to \mathbb{R}$, we write $M \leq N$ if $M(x) \leq N(x)$ for all $x \in X$ in the domain of $M$.

**Definition 7.1** (Exhaustion)**.** *We say that a sequence $(M_n)_{n\in\mathbb{N}}$ is an* exhaustion *of $\partial_* X$ if $M_n \leq M_{n+1}$ for all $n$ and for all Morse rays $\gamma : [0, \infty) \to X$ starting at $e$ we have that $\gamma$ is $M_n$–Morse for some $n$.*

Observe that the Morse boundary $\partial_* X$ is $\sigma$ compact if and only if there exists an exhaustion of $\partial_* X$ by [6, Lemma 2.1] and [7, Lemma 4.1].

**Lemma 7.2.** *Let $M$ be a Morse gauge and $(\lambda, \kappa)$–quasi-geodesic constants, there exists a Morse gauge $M'$ only depending on $M, \lambda$ and $\kappa$ such that the following holds:*

*(1) (Triangles) Let $\Delta$ be a triangle with (potentially unbounded) sides $\alpha, \beta, \gamma$ which are $(\lambda, \kappa)$–quasi-geodesics. If $\alpha$ and $\beta$ are $M$–Morse, then $\gamma$ is $M'$–Morse.*
*(2) (Equivalent quasi-geodesics) Let $\alpha, \beta$ be $(\lambda, \kappa)$–quasi-geodesics which are at bounded Hausdorff distance and which have the same starting point. If $\alpha$ is $M$–Morse, then $\beta$ is $M'$–Morse.*
*(3) (Bounded distance) Let $\alpha, \beta$ be $(\lambda, \kappa)$–quasi-geodesics such that the endpoints of $\beta$ are in the $\kappa$–neighbourhood of $\alpha$. If $\alpha$ is $M$–Morse, then $\beta$ is $M'$–Morse.*

*(4) (Concatenation) Let $\alpha$ and $\beta$ be $M$–Morse $(\lambda,\kappa)$–quasi-geodesics such that their concatenation $\gamma = \alpha * \beta$ is a $(\lambda,\kappa)$–quasi-geodesic. Then $\gamma$ is $M'$–Morse.*

7.1. **Weak Morse local-to-global.** We start by establishing some basic properties about weak Morse local-to-global spaces.

**Setup:** In this section we let $(X, \mathrm{d})$ denote a proper geodesic metric space which satisfies the weak MLTG and $G$ be a group acting coboundedly on $X$. Further, let $r \geq 0$ be the constant such that for every $x \in X$ we have $\mathcal{N}_r(G \cdot x) = X$.

The following is a version of Lemma 7.21 about Morse triangles. Specifically, it deals with the case when the sides are only locally Morse in the case of a weak Morse local-to-global space. As we only need this in a very specific setting, the statement and the proofs are specialized accordingly. However, the proof can be easily adapted to other, more general, settings.

**Lemma 7.3.** *Let $M, N$ be Morse gauges and $(\lambda,\kappa)$–quasi-geodesic constants. There exists a Morse gauge $N'$, a constant $\delta$ and a function $\tau : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that for every constant $L$ the following holds. Let $\eta$ be an $N$–Morse geodesic segment with endpoints $x, z$, and let $\gamma$ be a $(\lambda,\kappa)$–quasi-geodesic segment which is $\tau(L)$–locally $M$–Morse with endpoints $u$ and $v$. If $\mathrm{d}(u,z) \leq r$, then for any $y \in \eta$ the geodesic $[y, v]$ is $L$-locally $N'$–Morse and contained in the $\delta$–neighbourhood of $\eta|_{yz} \cup \gamma$.*

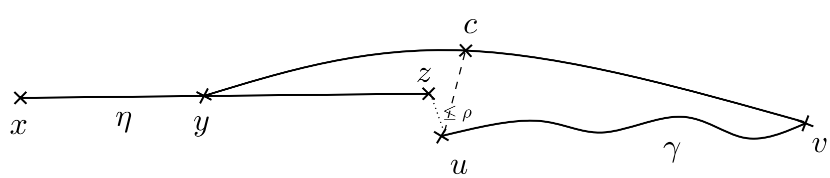


Figure 6. Proof of Lemma 7.3

*Proof.* Let $z$ be a closest point to $u$ on $[y, v]$. By Lemma 3.1, $\mathfrak{p}_1 = [u, z] * [y, v]|_{zv}$ is a $(3, 0)$–quasi-geodesic. By the weak MLTG property, up to choosing $\tau(L)$ large enough, we can guarantee that $\mathfrak{p}_1$ lies in a uniform neighbourood of $\gamma$. In particular, it is $L$–locally Morse, where the constants can be determined in terms of $L$ and the local Morse gauge of $\gamma$. Now consider the concatenation $\mathfrak{p}_2 = [y, v]|_{yc} * [c, u] * [u, z]$. It is a $(3, r)$–quasi-geodesic. Since $\eta$ is Morse, we have that $\mathfrak{p}_2$ lies in a uniform neighbourhood of $\eta$ and it is Morse, where the Morse gauge depends on $r$. Thus, $[y, b]$ is the concatenation of two geodesics which are both $L$–locally Morse. In particular, $[y, v]$ is $L$–locally Morse, for a worse Morse gauge. Moreover, since $\mathfrak{p}_1$ and $\mathfrak{p}_2$ are both in a uniform neighbourhood of $\eta|_{yz} \cup \gamma$, so is $[y, v]$. $\square$

**Lemma 7.4.** *Let $M, N$ be Morse gauges and $(\lambda,\kappa)$–quasi-geodesic constants. There exist Morse gauges $N'$ and $N_{\mathrm{not}}$, functions $f : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ and $g : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ and quasi-geodesic constants $(\lambda', \kappa')$ such that for every constant $L$ the following holds. For $i = 1, 2$, let $\eta_i$ be an $N$–Morse geodesic segment with endpoints $x_i, z_i$, and let $y_i$ be a point on $\eta_i$. Let $\gamma$ be a $(\lambda,\kappa)$–quasi-geodesic segment which is $g(L)$–locally $M$–Morse but not $N_{\mathrm{not}}$–Morse and whose endpoints $u$ and $v$ satisfy*

$$\mathrm{d}(z_1, u) \leq r \quad \textit{and} \quad \mathrm{d}(v, x_2) \leq r,$$
$$\mathrm{d}(y_1, z_1) \geq f(\mathrm{d}(u,v), L) \quad \textit{and} \quad \mathrm{d}(x_2, y_2) \geq f(\mathrm{d}(u,v), L).$$

*Then the path* $\mathfrak{p} = \eta_1|_{x_1y_1} * [y_1, y_2] * \eta_2|_{y_2z_2}$ *is a* $(\lambda', \kappa')$*–quasi-geodesic which is* $L$*-locally* $N'$*–Morse. Moreover, if* $\mathfrak{p}$ *is an* $\tilde{N}$*–Morse* $(\lambda'', \kappa'')$*–quasi-geodesic, then* $\gamma$ *is* $\tilde{N}'$*–Morse, where* $\tilde{N}'$ *only depends on* $\tilde{N}, N, \lambda, \kappa$ *and* $(\lambda'', \kappa'')$*.*

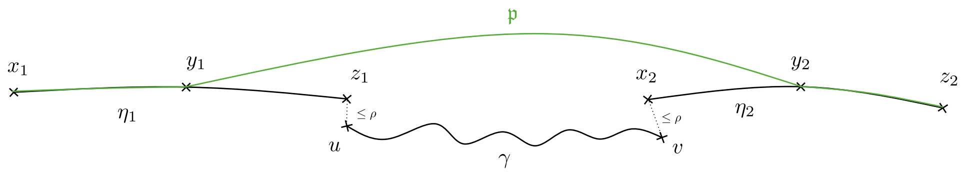


FIGURE 7. Depiction of Lemma 7.4

*Proof.* Applying Lemma 7.3 first to $[y_1, v]$ and then to $[y_1, y_2]$ yields the existence of a function $g$, a Morse gauge $N''$ and a constant $\delta$ such that if $\gamma$ is $g(L)$–locally $M$–Morse, then $[y_1, y_2]$ is $L$-locally $N''$–Morse and $[y_1, y_2]$ is contained in the $\delta$–neighbourhood of $\eta_1|_{y_1z_1} \cup \gamma \cup \eta_2|_{x_2y_2}$.

If $\mathrm{d}(\eta_1, \eta_2) \le 2\delta$, then applying Lemma 7.21 about Morse triangles multiple times yields a Morse gauge $N_0$ depending only on $N, \lambda, \kappa, r$ and $\delta$ such that $\gamma$ is $N_0$–Morse. Defining $N_{\mathrm{not}} = N_0$ yields that $\mathrm{d}(\eta_1, \eta_2) > 2\delta$.

Next we show that $\mathfrak{p}' = \eta_1|_{x_1y_1} * [y_1, y_2]$ is $L$–locally an $N'$–Morse $(\lambda', \kappa')$–quasi-geodesic for a Morse gauge $N'$ and quasi-geodesic constants $(\lambda', \kappa')$ which we will determine below. Let $y$ be the first point on $[y_1, y_2]$ which is in the closed $\delta$–neighbourhood of $\gamma \cup \eta_2$. By continuity, $y$ is in the closed $\delta$–neighbourhood of $\eta_1|_{y_1z_1}$. Since $\mathrm{d}(\eta_1, \eta_2) > 2\delta$, $y$ is not in the closed $\delta$–neighbourhood of $\eta_2$ and hence in the closed $\delta$–neighbourhood of $\gamma$. We now proceed to bound $\mathrm{d}(y, y_1)$ from below. By the triangle inequality, we have that

$$\begin{aligned}\mathrm{d}(y, y_1) &\ge \mathrm{d}(y_1, \gamma) - \delta,\\ &\ge \mathrm{d}(y_1, z_1) - \lambda(\lambda \mathrm{d}(u, v) + \kappa) - \kappa - \delta.\end{aligned}$$

Thus, for large enough $f$, we have that $\mathrm{d}(y, y_1) \ge L$. Define $(\lambda', \kappa') = (1, 2\delta)$. To prove that $\mathfrak{p}'$ is $L$-locally a $(\lambda', \kappa')$–quasi-geodesic it suffices to show that $\mathrm{d}(\mathfrak{p}'(s), \mathfrak{p}'(t)) \ge |s-t| - 2\delta$ for all $s, t$ with $\mathfrak{p}'(s) \in \eta_1|_{x_1y_1}$ and $\mathfrak{p}'(t) \in [y_1, y_2]$ with $\mathrm{d}(\mathfrak{p}'(t), y_1) \le L$. Let $s, t$ be such constants. Since $\mathrm{d}(y, y_1) \ge L$, there exists $c$ on $\eta_1|_{y_1z_1}$ with $\mathrm{d}(\mathfrak{p}'(t), c) \le \delta$ and hence $\mathrm{d}(y_1, c) \ge \mathrm{d}(y_1, \mathfrak{p}'(t)) - \delta$. Observe that $|s-t| = \mathrm{d}(\mathfrak{p}'(s), y_1) + \mathrm{d}(y_1, \mathfrak{p}'(t))$, implying that $\mathrm{d}(\mathfrak{p}'(s), \mathfrak{p}'(t)) \ge \mathrm{d}(\mathfrak{p}'(s), c) - \delta \ge |s-t| - 2\delta$. Hence $\mathfrak{p}'$ is indeed $L$–locally a $(\lambda', \kappa')$–quasi-geodesic. Furthermore, $[y_1, y_2]$ is $L$-locally $N''$–Morse and $\eta_1|_{x_1y_1}$ is $N$–Morse. Thus, by Lemma 7.24 about concatenations there exists a Morse gauge $N'$ only depending on $N, N'', \lambda$ and $\kappa$ such that $\mathfrak{p}'$ is $L$-locally $N'$–Morse.

Analogously, we can show that $[y_1, y_2] * \eta_2|_{y_2z_2}$ is $L$–locally an $N'$–Morse $(\lambda', \kappa')$–quasi-geodesic. Lastly, since $\mathrm{d}(y_1, y_2) \ge L$ this shows that $\mathfrak{p}$ as a whole is $L$–locally an $N'$–Morse $(\lambda', \kappa')$–quasi-geodesic.

It remains to prove the moreover part of the statement, which follows from repeatedly applying Lemma 7.21 about Morse triangles. □

**Lemma 7.5.** *Let* $M$ *and* $N$ *be Morse gauges and let* $(\lambda, \kappa)$ *be quasi-geodesic constants. Then there exists a Morse gauge* $N_{\mathrm{not}}$*, a constant* $L_{\min}$*, and a map* $\Phi$

*between Morse gauges such that the following holds. Suppose there exists an $N$–Morse geodesic ray $\eta$. Then for each sequence $\{\gamma_i\}$ of $(\lambda, \kappa)$–quasi-geodesic segments which are $L_i$-locally $M$–Morse but not $N_{\text{not}}$–Morse and with $L_i \geq L_{\min}$ and $\lim_{i\to\infty} L_i = \infty$, there exists a Morse geodesic ray $\zeta$ such that for all Morse gauges $\tilde{N}$, if $\zeta$ is $\tilde{N}$–Morse, then $\gamma_i$ is $\Phi(\tilde{N})$–Morse.*

*Proof.* Let $N', N_{\text{not}}, (\lambda', \kappa')$ and $f, g$ be the Morse gauges, quasi-geodesic constants and functions from Lemma 7.4 applied to the Morse gauges $M, N$ and quasi-geodesic constants $(\lambda, \kappa)$. By potentially increasing $N'$, we may assume that $N' \geq N$. Let $L_{\text{quasi}}, \lambda'', \kappa''$ be constants such that every $L_{\text{quasi}}$-locally $N'$–Morse $(\lambda', \kappa')$–quasi-geodesic is a $(\lambda'', \kappa'')$–quasi-geodesic. Let $L_{\min} = g(L_{\text{quasi}})$. For every quasi-geodesic pair $(Q, q)$, let $L_{Q,q}$ and $M(Q, q)$ be constants such that every $L_{Q,q}$–locally $N'$–Morse $(\lambda', \kappa')$–quasi-geodesic is a $(M(Q, q), Q, q)$–Morse $(\lambda'', \kappa'')$–quasi-geodesic.

Let $(\gamma_i)_i$ be a sequence as in the statement. For every $i \geq 1$ let $D_i = f(\mathrm{d}(u_i, v_i), L_i)$, where $u_i$ and $v_i$ are the endpoints of $\gamma_i$.

Define $\eta'_1 = \eta$ and let $x_1 = y_1$ be the starting point of $\eta'_1$. Let $z_1$ be a point on $\eta'_1$ such that $\mathrm{d}(y_1, z_1) \geq D_1$. Define $\eta_1$ as $\eta'_1|_{y_1 z_1}$. Lastly, define $\gamma'_1$ as a translate of $\gamma_1$ with endpoints $u'_1, v'_1$ such that $\mathrm{d}(u'_1, z_1) \leq r$. For $i \geq 1$ inductively define the following, which is depicted in Figure 8.

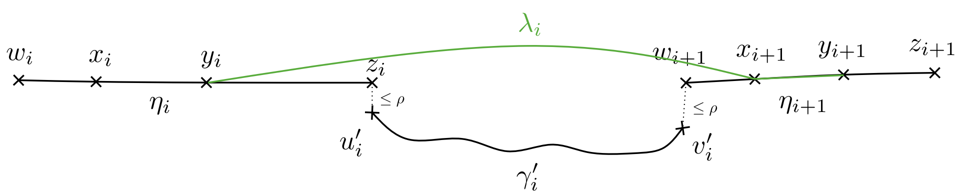


FIGURE 8. Proof of Lemma 7.5

- $\eta'_{i+1}$ as a translate of $\eta$ starting in the $r$–neighbourhood of $v'_i$
- $w_{i+1}$ as the starting point of $\eta'_{i+1}$
- $x_{i+1}$ as a point on $\eta'_{i+1}$ with $\mathrm{d}(w_{i+1}, x_{i+1}) = D_i$.
- $y_{i+1}$ as a point on $\eta'_{i+1}$ with $\mathrm{d}(w_{i+1}, y_{i+1}) = D_i + L_{\text{quasi}} + i$, and hence $\mathrm{d}(x_{i+1}, y_{i+1}) = L_{\text{quasi}} + i$.
- $z_{i+1}$ as a point on $\eta'_{i+1}$ with $\mathrm{d}(w_{i+1}, z_{i+1}) = D_i + L_{\text{quasi}} + i + D_{i+1}$ and hence $\mathrm{d}(y_{i+1}, z_{i+1}) = D_{i+1}$.
- $\eta_{i+1}$ as $\eta'_{i+1}|_{w_{i+1} z_{i+1}}$.
- $\gamma'_{i+1}$ as a translate of $\gamma_{i+1}$ with endpoints $u'_{i+1}, v'_{i+1}$ and such that $\mathrm{d}(u'_{i+1}, z_{i+1}) \leq r$.

For $i \geq 1$ define $\zeta_i = [y_i, x_{i+1}] * [x_{i+1}, y_{i+1}]$. And for each $i$ define $\zeta'_i$ as the infinite concatenation

$$\zeta'_i = \zeta_i * \zeta_{i+1} * \zeta_{i+2} * \dots$$

Next we show that $\zeta'_1$ is a quasi-geodesic. Indeed, since $L_i \geq g(L_{\text{quasi}})$ for all $i$, the path $[x_i, y_i] * [y_i, x_{i+1}] * [x_{i+1}, y_{i+1}]$ is an $L_{\text{quasi}}$–locally $N'$–Morse $(\lambda', \kappa')$–quasi-geodesic by Lemma 7.4. Since $\mathrm{d}(x_i, y_i) \geq L_{\text{quasi}}$, the above implies that $\zeta'_1$ is $L_{\text{quasi}}$–locally an $N'$–Morse $(\lambda', \kappa')$–quasi-geodesic. By the choice of $L_{\text{quasi}}$, $\zeta'_1$ is a $(\lambda'', \kappa'')$–quasi-geodesic.

Now we show that $\zeta_1'$ is $N_0$–Morse for a Morse gauge $N_0$ which we are about to construct. Let $(Q,q)$ be a quasi-geodesic pair. Since $\lim_{i\to\infty} L_i = \infty$, there exists $i_{Q,q} \geq L_{Q,q}$ such that for all $i \geq i_{Q,q}$ we have that $L_i \geq g(L_{Q,q})$. Hence by Lemma 7.4, $[x_i, y_i] * [y_i, x_{i+1}] * [x_{i+1}, y_{i+1}]$ is $L_{Q,q}$–locally an $N'$–Morse $(\lambda', \kappa')$–quasi-geodesic for all $i \geq i_{Q,q}$. Since $i_{Q,q} \geq L_{Q,q}$ and $\mathrm{d}(x_i, y_i) \geq i$ we have that $\zeta'_{i_{Q,q}}$ is also $L_{Q,q}$–locally an $N'$–Morse $(\lambda', \kappa')$–quasi-geodesic. Hence, by the definition of $L_{Q,q}$, $\zeta'_{i_{Q,q}}$ is $(M_{Q,q}, Q, q)$–Morse. Consider $\mathfrak{p}_{Q,q} = \zeta_1 * \ldots * \zeta_{i_{Q,q}-1}$. We have that $\zeta_1' = \mathfrak{p}_{Q,q} * \zeta'_{i_{Q,q}}$. Further, since $\mathfrak{p}_{Q,q}$ is a finite subsegment of $\zeta_1'$, there exists a constant $N_{Q,q}$ such that $\mathfrak{p}$ is $(N_{Q,q}, Q, q)$–Morse. Define

$$N_0(Q/3, q) = \max\{N_{Q,q}, M_{Q,q}\}.$$

**Claim 7.** *The quasi-geodesic $\zeta_1'$ is $N_0$–Morse.*

*Proof of Claim 7.* Let $\xi$ be a $(Q/3, q)$–quasi-geodesic with endpoints $a$ and $b$ on $\zeta_1'$. If both its endpoints are on $\mathfrak{p}_{Q,q}$, then, since $\mathfrak{p}_{Q,q}$ is $(N_{Q,q}, Q, q)$–Morse $\xi$ is contained in the $N_{Q,q}$–neighbourhood of $\zeta_1'|_{ab}$. If both endpoints $a$ and $b$ are contained in $\zeta'_{i_{Q,q}}$, then, since $\zeta'_{i_{Q,q}}$ is $(M_{Q,q}, Q, q)$–Morse, $\xi$ is contained in the $M_{Q,q}$ neighbourhood of $\zeta_1'|_{ab}$. It remains to show that $\xi$ stays in the $N_0(Q/3, q)$–neighbourhood of $\zeta_1|_{[a,b]}$ if $a$ lies on $\mathfrak{p}_{Q,q}$ and $b$ lies on $\zeta'_{i_{Q,q}}$. Let $c$ be the endpoint of $\mathfrak{p}_{Q,q}$ (and hence the starting point of $\zeta'_{i_{Q,q}}$). Let $c'$ be a closest point on $\xi$ to $c$. By Lemma 3.1 the paths $\xi|_{ac'} * [c', c]$ and $[c, c'] * \xi|_{c'b}$ are $(Q, q)$–quasi-geodesics. By the arguments above, they are contained in the $N_{Q,q}$ and $M_{Q,q}$–neighbourhood of $\zeta_1'|_{ac}$ and $\zeta_1'|_{cb}$ respectively. Since $N_0(Q/3, q) = \max\{N_{Q,q}, M_{Q,q}\}$, the statement follows ■

We have showed that $\zeta_1'$ is a Morse $(\lambda'', \kappa'')$–quasi-geodesic, hence there exists a geodesic $\zeta$ with the same starting point which is at bounded Hausdorff distance from $\zeta_1'$. Note that $(\lambda'', \kappa'')$ only depend on $X, M, \lambda, \kappa$ and not the quasi-geodesics $\gamma_i$. Assume that $\zeta$ is $\tilde{N}$–Morse for some Morse gauge $\tilde{N}$. By Lemma 7.22, $\zeta_1'$ is $\tilde{N}''$–Morse, where $\tilde{N}''$ only depends on $\tilde{N}, \lambda''$ and $\kappa''$. Since $[x_i, y_i] * [y_i, x_{i+1}] * [x_{i+1}, y_i]$ is a subsegment of $\zeta_1'$, it is also $\tilde{N}''$–Morse. By Lemma 7.4, $\gamma_i$ is $\tilde{N}'$–Morse, where $\tilde{N}'$ only depends on $\tilde{N}'', N, M, X, \lambda$ and $\kappa$, which concludes the proof. □

Now we are ready to prove Theorem 1.5.

*Proof of Theorem 1.5.* We assume that $X$ satisfies the weak MLGT but not the MLTG. Observe that any space with empty Morse boundary satisfies the MLTG, thus $\partial_* X$ is not empty, and there exists a ray $\eta$ which is $N$–Morse for some Morse gauge $N$. We want to show that $\partial_* X$ is not $\sigma$-compact. More precisely, we will prove that $X$ does not have an exhaustion.

Assume by contradiction that $X$ has an exhaustion $(M_n)_{n\in\mathbb{N}}$. Let $(M, \lambda', \kappa')$ be a triple that fails the Morse local-to-global property. Since $X$ satisfies the weak MLTG, there exist constants $L_0, \lambda, \kappa$ such that all $L_0$–locally $M$–Morse $(\lambda, \kappa')$–quasi-geodesics are $(\lambda, \kappa)$–quasi-geodesics. Thus we can apply Lemma 7.5 to the Morse gauges $M$ and $N$ and the quasi-geodesic constants $(\lambda, \kappa)$ to get a Morse gauge $N_{\text{not}}$, constant $L_{\min}$ and a map $\Phi$ between Morse gauges.

For each $i \geq 1$, define $L_i = \max\{i, L_{\min}, L_0\}$. Since $(M, \lambda', \kappa')$ fails the Morse local-to-global property, there exists a path $\gamma_i$ which is $L_i$–locally an $M$–Morse $(\lambda', \kappa')$-quasi-geodesic but not a $\max\{N_{\text{not}}, \Phi(M_i)\}$–Morse $(\lambda, \kappa)$–quasi-geodesic. Since $L_i \geq L_0$, $\gamma_i$ is a $(\lambda, \kappa)$–quasi-geodesic and hence the failure of the MLTG property is that $\gamma_i$ is not $\Phi(M_i)$–Morse. Moreover, by potentially replacing $\gamma_i$ with a subsegment, we can assume that $\gamma_i$ is finite. Indeed, if all finite subsegments of $\gamma_i$ were $\Phi(M_i)$–Morse, $\gamma_i$ itself would be $\Phi(M_i)$–Morse.

Let $\zeta$ be the Morse geodesic obtained from Lemma 7.5 applied to the sequence $\{\gamma_i\}$. Since $\zeta$ is Morse and $(M_n)_{n\in\mathbb{N}}$ is an exhaustion, there exists a Morse gauge $M_n$ such that $\zeta$ is $M_n$–Morse, implying that $\gamma_n$ is $\Phi(M_n)$–Morse. However, we precisely chose $\gamma_n$ to not be $\Phi(M_n)$–Morse, which is a contradiction. The statement follows. □

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